



# Relations among Density-based Equicontinuities

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**Abstract:** Equicontinuity is a regularity condition for characterizing global orbital stability in topological dynamical systems (t.d.s.). This study investigates the relationship between four kinds of density-based equicontinuity in t.d.s., and examines their inhibitory effects on chaos. Under the framework of minimality, density-equicontinuity, mean equicontinuity, density- $t$ -equicontinuity, and almost density-equicontinuity are obtained to be equivalent to each other. And density-equicontinuity is incompatible with chaos.

**Keywords:** density-equicontinuity, minimal systems, chaos, topological dynamical systems.

## INTRODUCTION

The characterization of orbital stability and complex behaviors has always been a core issue in dynamical systems. A topological dynamical systems (t.d.s.) consists of the triple  $(X, d, T)$ , in which  $X$  denotes a compact metric space endowed with the metric  $d$ , and  $T: X \rightarrow X$  represents a continuous self-map. This definition aligns with that of the t.d.s. presented in [1]. Birkhoff's classic work laid the theoretical foundation for this field and established the basic paradigm for the study of dynamics of continuous maps on compact spaces ([2]).

As a key notion for depicting stability features, equicontinuity was comprehensively elaborated by Gottschalk [1]. Specifically, for any given positive real number  $\varepsilon$ , there exists a corresponding positive real number  $\delta$  such that if  $d(x, y) < \delta$ , then  $d(T^n x, T^n y) < \varepsilon$  for all non-negative integers  $n$ . This strong constraint ensures long-term stability of orbits but fails to accommodate systems with local fluctuations in practical scenarios.

To break through this limitation, Fomin [3] proposed mean-L-stability in 1951, allowing orbital deviations to only appear in 'asymptotically sparse' time sets. Subsequent studies confirmed that this concept is equivalent to the mean equicontinuity, which is proposed by Li [4] and is defined via the Cesàro mean limit. A map  $T: X \rightarrow X$  is mean equicontinuous if for any  $\varepsilon > 0$ , a  $\delta > 0$  exists such that  $d(x, y) < \delta$  implies

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) < \varepsilon.$$

This progress provided an effective tool for studying 'statistically stable' systems, while Qiu [5] further expanded the research dimension by extending mean equicontinuity results from minimal to general t.d.s., and establishing its equivalent characterizations and a framework for maximal mean equicontinuous factors. Li [6] summarized recent advances,

highlighting mean equicontinuity's links to discrete spectrum, topological isomorphy, bounded complexity, and its number-theoretic applications (e.g., Sarnak conjectures). Later on, the research team explored two mean-based forms of equicontinuity and sensitivity, which revealed that any t.d.s. can be embedded into some almost equicontinuous in the mean system ([7]). Such insights not only affirm mean equicontinuity's theoretical significance but also motivate the generalization of 'mean-based' stability to 'density-based' concepts—the central theme of this work.

In the analysis of stability and complexity, Xiong [8] generalized the sensitive reliance on initial states and analyzed the orbital complexity of transitive systems, while Ye [9] established a method for determining sensitivity and proved that transitive systems have maximal almost equicontinuous factors. Li [4] explored almost mean equicontinuity, revealing the intrinsic connection between partial stability and system complexity.

Building on this, Li [10] introduced density-equicontinuity and Banach density-equicontinuity, proving their equivalence (Theorem 1.2). Regarding mean equicontinuity, Proposition 3.1 in [10] indicates that a system with mean equicontinuous is equivalent to a density- $t$ -equicontinuous system for each  $t \in [0,1)$ .

Liu [11] proposed two density-equicontinuity notions associated with a specified Følner sequence (a key tool for amenable group actions), together with Banach density-equicontinuity adapted to amenable group actions. He also established connections between these concepts and Banach mean-equicontinuity as well as Banach density- $t$ -equicontinuity, providing support for the statistical stability analysis of generalized systems.

Building on the aforementioned literature, this paper explores density-based equicontinuity and chaoticity. The following two key questions are focused on. (1) What relationships between density-equicontinuity, mean equicontinuity, density- $t$ -equicontinuity, and almost density-equicontinuity? (2) How do these density-based equicontinuity interact to influence chaotic behavior? To address these questions, Section 3 focuses on the relationship between four kinds of density-based equicontinuity, and Section 4 explores the inhibitory effect of density-based equicontinuity on chaos.

## PRELIMINARIES

In this paper,  $\mathbb{N}$  refers to the collection of natural numbers  $\{1,2,3,\dots\}$ , while  $\mathbb{Z}_+$  stands for the set consisting of non-negative integers  $\{0,1,2,\dots\}$ . Before introducing the concepts of density-based equicontinuity, the relevant definitions of density are necessary recalled. For any subset  $S \subseteq \mathbb{Z}_+$ , the upper density  $\bar{D}(S)$  and lower density  $\underline{D}(S)$  are defined as follows.

$$\bar{D}(S) = \limsup_{n \rightarrow \infty} \frac{\#\{S \cap \{0,1,\dots,n-1\}\}}{n}$$

and

$$\underline{D}(S) = \liminf_{n \rightarrow \infty} \frac{\#\{S \cap \{0,1,\dots,n-1\}\}}{n},$$

correspondingly. Where, the symbol  $\#\{\cdot\}$  stands for the cardinality of a given set.

The lower density denotes the minimum of asymptotic densities over all natural number intervals, representing the slowest long-term growth rate of  $S$ . By contrast, the upper density denotes the maximum of asymptotic densities over all natural number intervals, representing the fastest long-term growth rate of  $S$ .

If the upper density equals the lower density, i.e.,  $\bar{D}(S) = \underline{D}(S)$ , this shared value is referred to as the density of set  $S$ , symbolized as  $D(S)$ . Specifically,

$$D(S) = \lim_{n \rightarrow \infty} \frac{\#\{S \cap \{0, 1, \dots, n-1\}\}}{n},$$

which provided the limit exists. This concept is central in ergodic theory and combinatorics, as it characterizes subsets with uniform density across all sufficiently long intervals.

To support the subsequent analysis of equicontinuity, two key properties of upper density subadditivity and translation invariance are critical, as they lay a quantitative foundation for quantifying orbit separation sets.

The subadditivity of upper density mean that, for any subsets  $A, B \subset \mathbb{Z}_+$ ,

$$\bar{D}(A \cup B) \leq \bar{D}(A) + \bar{D}(B).$$

The translation invariance of upper density mean that, for any subset  $S \subset \mathbb{Z}_+$  and any  $m \in \mathbb{Z}_+$ ,  $\bar{D}(S) = \bar{D}(S+m)$ , where  $S+m = \{s+m : s \in S\}$  (called the translation of  $S$  by  $m$ ).

The aforementioned density tools provide a quantitative basis for characterizing the 'frequency constraints' of orbit separation, and serve as the core support for the subsequent definition of equicontinuity.

**Definition 2.1:** ([10]) A t.d.s.  $(X, d, T)$  is said to be density-equicontinuous if for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for any  $x, y \in X$  satisfying  $d(x, y) < \delta$ , the set  $F := \{k \in \mathbb{Z}_+ : d(T^k x, T^k y) > \varepsilon\}$  has upper density  $\bar{D}(F) = 0$ .

**Definition 2.2:** ([10]) For a t.d.s.  $(X, d, T)$ , the system is said to be density- $t$ -equicontinuous with  $t \in [0, 1]$  if for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for any  $x, y \in X$  with  $d(x, y) < \delta$ , the upper density of the orbit separation set

$$\bar{D}(F) = \{k \in \mathbb{Z}_+ : d(T^k x, T^k y) > \varepsilon\} \leq 1 - t.$$

Specifically, if  $t = 1$ , the condition reduces to  $\bar{D}(F) = 0$ , the system is simply called density-equicontinuous.

Such a zero-upper density orbit separation set  $F$  is called an asymptotically sparse set, and these two terms are used alternately in this paper.

This means orbits of sufficiently close initial points stay  $\varepsilon$ -close for almost all iterations, with separations forming a zero-upper-density set.

**Definition 2.3:** ([10]) Let  $(X, d, T)$  be a t.d.s. A point  $x \in X$  is called a density-equicontinuous point if for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that, for  $U = B(x, \delta) \subset X$ , any  $y, z \in U$ , the upper density of the orbit separation set

$$\bar{D}(F_{y,z}) = \{n \in \mathbb{Z}_+ : d(T^n y, T^n z) > \varepsilon\} = 0$$

Equivalently,  $x \in X$  is a density-equicontinuous point if for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for any  $y \in X$  with  $d(x, y) < \delta$ , the upper density of the set  $F_{x,y} = \{n \in \mathbb{Z}_+ : d(T^n x, T^n y) > \varepsilon\}$  is 0.

**Definition 2.4:** ([10]) A t.d.s.  $(X, d, T)$  is called almost density-equicontinuous if there exists a density-equicontinuous transitive point  $x_0 \in X$ .

**Remark 2.1:** Density- $t$ -equicontinuity is a global quantitative constraint that applies to all point pairs. Almost density-equicontinuity is a local qualitative constraint restricted to the neighborhood of a transitive point, with zero upper density of separations.

**Definition 2.5:** ([4]) A t.d.s.  $(X, d, T)$  is referred to as mean equicontinuous if for each  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that whenever  $x, y \in X$  with  $d(x, y) < \delta$ , the following holds

$$\limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} d(T^i x, T^i y) < \varepsilon.$$

**Definition 2.6:** ([12]) A t.d.s. is said to be Li-Yorke chaotic if there exists an uncountable subset  $S \subseteq X$  that contains no periodic points, and for any distinct  $x, y \in S$ ,

$$\limsup_{k \rightarrow \infty} d(T^k(x), T^k(y)) > 0 \quad \text{and} \quad \liminf_{k \rightarrow \infty} d(T^k(x), T^k(y)) = 0.$$

In this case,  $(x, y)$  is called a Li-Yorke scrambled pair, and

$$LY(T) = \left\{ (x, y) \in S \mid \limsup_{k \rightarrow \infty} d(T^k(x), T^k(y)) > 0 \quad \text{and} \quad \liminf_{k \rightarrow \infty} d(T^k(x), T^k(y)) = 0 \right\}$$

is called a Li-Yorke scrambled set.

**Definition 2.7:** ([13]) A t.d.s. is said to be Devaney chaotic if it meets the three conditions below.

- (1) Topological transitivity. For any non-empty open subsets  $U, V \subset X$ , there exists a  $k \in \mathbb{Z}_+$  such that  $T^k(U) \cap V \neq \emptyset$ .
- (2) Dense periodic points. The set of periodic points is dense in  $X$ , where a periodic point  $p \in X$  means that there exists an  $m \geq 1$  such that  $T^m(p) = p$  and  $T^j(p) \neq p$  for all  $j < m$ . In other words, for any  $x \in X$  and  $\delta > 0$ , there exists a periodic point  $p \in X$  satisfying  $d(x, p) < \delta$ .
- (3) Sensitive dependence on initial conditions. There is a  $\gamma > 0$  such that for any  $x \in X$  and  $\delta > 0$ , there exists a  $y \in X$  with  $d(x, y) < \delta$  and some  $k \in \mathbb{Z}_+$  such that  $d(T^k x, T^k y) > \gamma$ .

Sensitivity is a core characteristic of Devaney chaos. In [14], Banks pointed out that (1) and (2) implies (3). The classical Auslander-Yorke dichotomy holds every transitive system as either equicontinuous or sensitive, with recent studies further extending and reexamining this claim ([15]). Specifically, [15] proposed a way to classify topologically transitive systems using equicontinuity pairs, and extended the classic Auslander-Yorke dichotomy to minimal systems. Additionally, the authors also identified a special transitive system with continuity pairs but no equicontinuity points, which further enriches the theoretical connection between equicontinuity and sensitivity.

### RELATIONS AMONG DENSITY-BASED EQUICONTINUITIES

Intuitively, density- $t$ -equicontinuity exhibits global stability, while almost density-equicontinuity restricts stability to the neighborhood of a transitive point without constraining behaviors in other regions of the space.

Existing studies have clarified their relationships with other kinds of equicontinuities. For example, Li [10] proves density- $t$ -equicontinuity ( $t \in [0, 1)$ ) is equivalent to mean equicontinuity, and demonstrates a dichotomy between almost density-equicontinuity and density-sensitivity in transitive systems.

Notably, Proposition 3.4(3) in [10] states that almost density-equicontinuous minimal systems are density-equicontinuous. A natural question arises: Is almost density-equicontinuity equivalent to density- $t$ -equicontinuity? To address this, an initial understanding of their relationship is first gained through Example 3.1 presented below.

**Example 3.1:** There exists a t.d.s. that is almost density-equicontinuous, density-equicontinuous, and satisfies density- $t$ -equicontinuity for all  $t \in [0, 1]$ .

**Proof** Let the state space  $X = [0, 1]$  be endowed with the circular metric

$$d(x, y) = \min\{|x - y|, 1 - |x - y|\}.$$

Define a continuous self-map  $T : X \rightarrow X$  as

$$T(x) = (x + \alpha) \pmod{1},$$

where  $\alpha = \sqrt{2} - 1 \approx 0.414$  and  $\alpha$  is an irrational number. The modulo operation satisfies continuity on the compact interval  $[0, 1]$ , so  $(X, d, T)$  forms a t.d.s. According to the basic properties of number theory, the sequence of fractional parts of multiples of the irrational number  $\alpha$ , denoted  $\{n\alpha \pmod{1}\}_{n=1}^{\infty}$ , is dense in  $[0, 1]$  ([16]). Therefore, the orbit of every point in  $X$  under map  $T$  is dense in  $X$ .

Take  $x_0 = 0.2 \in X$ , which is a transitive point. For any  $\varepsilon > 0$ , take  $\delta = \min\left\{\frac{\varepsilon}{2}, 0.01\right\}$  that is sufficiently small to ensure the neighborhood is fully contained in  $X$ , and define the neighborhood of  $x_0$  as  $U = B(x_0, \delta) = (x_0 - \delta, x_0 + \delta) \subset X$ . For any  $y, z \in U$  with  $d(y, z) < \delta$ ,  $T$  is an irrational rotation map and satisfies distance preservation. Thus, for all  $k \in \mathbb{Z}_+$ ,  $d(T^k y, T^k z) = d(y, z)$ . Since  $d(y, z) < \delta \leq \frac{\varepsilon}{2} < \varepsilon$ , then  $d(T^k y, T^k z) < \varepsilon$  ( $k \in \mathbb{Z}_+$ ). So,  $F_{y,z} = \{k \in \mathbb{Z}_+ \mid d(T^k y, T^k z) > \varepsilon\} = \emptyset$  and  $\bar{D}(F_{y,z}) = 0$ , which satisfies the requirement of almost density-equicontinuity. The following proves that the system satisfies density-equicontinuity.

By Definition 2.1, for any  $\varepsilon > 0$ , there exists a  $\delta > 0$ , for any  $x, y \in X$  satisfying  $d(x, y) < \delta$ , the set  $F = \{k \in \mathbb{Z}_+ \mid d(T^k x, T^k y) > \varepsilon\}$  fulfills  $\bar{D}(F) = 0$ .

Set  $\delta = \varepsilon$ . For all  $x, y \in X$  satisfying  $d(x, y) < \delta$ , since  $T$  is a globally distance-preserving rotation map, then  $d(T^k x, T^k y) = d(x, y)$  for every  $k \in \mathbb{Z}_+$ . Given  $d(x, y) < \delta = \varepsilon$ , it follows that  $d(T^k x, T^k y) < \varepsilon$  for every  $k \in \mathbb{Z}_+$ . Thus, the orbit separation set  $F_{x,y} = \emptyset$ , and so its upper density

$$\bar{D}(F) = 0.$$

So,  $(X, d, T)$  is density-equicontinuity. Furthermore, for all  $t \in [0, 1]$ , since  $\bar{D}(F) = 0$  and  $1 - t \geq 0$  for any  $t \in [0, 1]$ , density- $t$ -equicontinuity requires  $\bar{D}(F) \leq 1 - t$ . Therefore, the system satisfies density- $t$ -equicontinuity for all  $t \in [0, 1]$ . In conclusion, there exists a t.d.s.  $(X, d, T)$  is almost density-equicontinuous, density-equicontinuous, and satisfies density- $t$ -equicontinuity for all  $t \in [0, 1]$ .

Since chaotic systems (e.g., the tent map) behave differently from equicontinuous systems ([17]), it remains unclear whether such coexistence or equivalence of density-based equicontinuity notions holds only for equicontinuous systems or can be generalized. For example, is almost density-equicontinuity equivalent to density- $t$ -equicontinuity? To address this, we first establish Lemma 3.1.

**Lemma 3.1:** Let  $(X, d, T)$  denote a minimal system without proper invariant closed subsets. Let  $x_0 \in X$  be a transitive point of the system, then, for any  $x, y \in X$  and any  $\delta > 0$ , there exists some common iteration  $n \in \mathbb{Z}_+$  such that  $T^n x \in B(x_0, \delta)$  and  $T^n y \in B(x_0, \delta)$ .

Proof Fix a  $\delta_0 > 0$ , for all  $n \in \mathbb{Z}_+$ ,  $T^n x \in B(x_0, \delta_0)$  and  $T^n y \in B(x_0, \delta_0)$  never hold simultaneously. By De Morgan's laws, this assumption is equivalent to the statement that for any  $n \in \mathbb{Z}_+$ ,  $T^n y \in B(x_0, \delta_0)$  implies  $T^n x \notin B(x_0, \delta_0)$ . Conversely, if  $T^n x \in B(x_0, \delta_0)$ , then  $T^n y \notin B(x_0, \delta_0)$ . Define the set

$$Y = \overline{\{T^n y \mid y \in X, n \in \mathbb{Z}_+, T^n x \notin B(x_0, \delta_0)\}}.$$

Given that  $x$  serves as a transitive point within a minimal system, the orbit  $\text{orb}(x, T)$  exhibits dense in  $X$ . If  $T^n x$  always fell into  $B(x_0, \delta_0)$ , then  $\overline{\text{orb}(x, T)} \subseteq \overline{B(x_0, \delta_0)} \subsetneq X$ , contradicting transitivity. Thus, infinitely many  $n \in \mathbb{Z}_+$  exist such that  $T^n x \notin B(x_0, \delta_0)$ , so the set  $\{T^n y \mid y \in X, n \in \mathbb{Z}_+, T^n x \notin B(x_0, \delta_0)\}$  is nonempty. Thus its closure  $Y$  is also nonempty. By definition of closure,  $Y$  constitutes a closed subset within  $X$ .

Take any  $z \in Y$ , a sequence  $\{n_k\}_{k=1}^\infty \subseteq \mathbb{Z}_+$  exists such that  $T^{n_k} y \rightarrow z$  and  $T^{n_k} x \notin B(x_0, \delta_0)$ . Since  $T$  is continuous,  $T^{n_k+1} y \rightarrow T(z)$ . Suppose  $T^{n_k+1} x \in B(x_0, \delta_0)$ , then for sufficiently large  $k$ ,  $T^{n_k+1} y \in B(x_0, \delta_0)$ , contradicting the assumption. Therefore,  $T^{n_k+1} x \notin B(x_0, \delta_0)$ , meaning  $T^{n_k+1} y \in Y$ , and its limit  $T(z)$  lies in  $Y$ , so  $T(Y) \subseteq Y$ .

Lastly, it is shown that  $Y$  constitutes a proper subset within  $X$ . In fact, given that  $x_0$  serves as a transitive point, its orbit exhibits dense. Hence an  $m \in \mathbb{Z}_+$  exists such that  $T^m x_0 \in B(x_0, \delta_0)$ . If  $T^m x_0 \in Y$ , a sequence  $\{n_k\}_{k=1}^\infty$  exists such that  $T^{n_k} y \rightarrow T^m x_0$  and  $T^{n_k} x \notin B(x_0, \delta_0)$ . For sufficiently large  $k$ ,  $T^{n_k} y \in B(x_0, \delta_0)$ , and by the assumption,  $T^{n_k} x \notin B(x_0, \delta_0)$ . However,  $B(x_0, \delta_0)$  contains iteration points of  $x$  (for example,  $T^p x \in B(x_0, \delta_0)$ , and in this case  $n = p$  forces), contradicting the limit property that. Therefore,  $T^m x_0 \notin Y$ . Thus,  $Y \subset X$ , so  $Y$  is a proper invariant closed subset of  $X$ . This contradicts the minimal of  $X$ . Therefore, there exists an  $n$  such that  $T^n x \in B(x_0, \delta)$  and  $T^n y \in B(x_0, \delta)$ .

**Theorem 3.1:** In a minimal t.d.s.  $(X, d, T)$ , mean equicontinuity is equivalent to almost density-equicontinuity.

Proof (Necessity) Suppose minimal  $(X, d, T)$  exhibits mean equicontinuous, then for any  $\varepsilon > 0$ ,  $x \in D(X)$  if and only if a  $\delta > 0$  exists such that for any  $x, y \in X$  satisfying  $d(x, y) < \delta$ ,

$$\limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} d(T^i x, T^i y) < \varepsilon,$$

where  $D(X)$  be the set of equicontinuous points, i.e.,  $x \in D(X)$  if and only if for any  $\varepsilon > 0$ , there exists a  $\delta > 0$ , for any  $y \in X$  with  $d(x, y) < \delta$ ,  $\limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} d(T^i x, T^i y) < \varepsilon$ . As shown in Proposition 4.2 in [4],  $D(X)$  is a  $G_\delta$  subset of  $X$ .

Given that  $(X, d, T)$  is minimal, the Baire category theorem yields that the collection of transitive points  $Trans(X)$  constitutes a dense  $G_\delta$  subset within  $X$ . The intersection of two dense  $G_\delta$  subsets is nonempty, so  $Trans(X) \cap D(X) \neq \emptyset$ .

Take  $x_0 \in Trans(X) \cap D(X)$ . Given that  $x_0$  serves as an equicontinuous point, for any  $\varepsilon > 0$ , a  $\delta > 0$  exists such that  $d(x_0, y) < \delta$  implies

$$\limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} d(T^i x_0, T^i y) < \varepsilon.$$

By the relations between Cesàro mean limits and upper density, if  $\limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} d(T^i x_0, T^i y) < \varepsilon$ , then the upper density  $\bar{D}(\{k \in \mathbb{Z}_+ : d(T^k x_0, T^k y) > \varepsilon\}) = 0$ . This indicates  $x_0$  is a density-equicontinuous transitive point, so the system is almost density-equicontinuous.

(Sufficiency) Suppose minimal  $(X, d, T)$  exhibits almost density-equicontinuous, which admits a density-equicontinuous transitive point  $x_0$ . For any  $x, y \in X$ , due to the system's minimality and transitivity, for any given  $\delta_1 > 0$ , an  $n \in \mathbb{Z}_+$  exists such that  $T^n x \in B(x_0, \frac{\delta_1}{2})$  and  $T^n y \in B(x_0, \frac{\delta_1}{2})$ .

Let  $x' = T^n x$  and  $y' = T^n y$ , then  $d(x', y') \leq d(x', x_0) + d(x_0, y') < \delta_1$ . Since  $x_0$  is density-equicontinuous, a  $\delta_2 > 0$  exists such that  $d(x_0, z) < \delta_2$  ensures

$$\bar{D}(\{k \in \mathbb{Z}_+ : d(T^k x_0, T^k z) > \frac{\varepsilon}{2}\}) = 0.$$

Take  $\delta' = \min\{\delta_1, \delta_2\}$ . If  $d(x, y) < \delta'$ , then  $d(x', y') < \delta'$ . More importantly,

$$d(x', x_0) < \frac{\delta_1}{2} < \delta_1 \leq \delta' < \delta_2 \quad \text{and} \quad d(y', x_0) < \frac{\delta_1}{2} < \delta_1 \leq \delta' < \delta_2.$$

So,  $\bar{D}(\{k : d(T^k x_0, T^k x') > \frac{\varepsilon}{2}\}) = 0$  and  $\bar{D}(\{k : d(T^k x_0, T^k y') > \frac{\varepsilon}{2}\}) = 0$ .

According to the triangle inequality,  $d(T^k x', T^k y') \leq d(T^k x', T^k x_0) + d(T^k x_0, T^k y')$ , then

$$\{k : d(T^k x', T^k y') > \varepsilon\} \subseteq \{k : d(T^k x', T^k x_0) > \frac{\varepsilon}{2}\} \cup \{k : d(T^k x_0, T^k y') > \frac{\varepsilon}{2}\}.$$

The subadditivity of the upper density,

$$\bar{D}(\{k : d(T^k x', T^k y') > \varepsilon\}) \leq \bar{D}(A) + \bar{D}(B) = 0.$$

By the iterative property and variable substitution,  $d(T^m x, T^m y) = d(T^{m-n} x', T^{m-n} y')$ . Thus,  $\bar{D}(\{m : d(T^m x, T^m y) > \varepsilon\}) = 0$ .

A standard result describes the relationship between zero upper density and the Cesàro mean limit. If  $\bar{D}(\{k : d(T^k x, T^k y) > \varepsilon\}) = 0$ , then  $\limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} d(T^i x, T^i y) \leq \varepsilon$ . Thus,

for any  $\varepsilon > 0$ , a  $\delta > 0$  exists such that  $d(x, y) < \delta$  ensures  $\limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} d(T^i x, T^i y) < \varepsilon$ .

Hence, the system exhibits mean equicontinuous. To conclude, for minimal t.d.s., almost density-equicontinuity coincides with mean equicontinuity.

By Proposition 3.1 in [10], a t.d.s. exhibits mean equicontinuity if and only if it is density- $t$ -equicontinuous for every  $t \in [0, 1)$ . Combining with Theorem 3.1, the following Theorem 3.2 can be obtained.

**Theorem 3.2:** For a minimal t.d.s., density- $t$ -equicontinuity ( $t \in [0, 1)$ ) holds if and only if the system is almost density-equicontinuous.

In addition, it can be proved that  $t = 1$  is also true (Theorem 3.3).

**Theorem 3.3:** A minimal t.d.s.  $(X, d, T)$  is density-equicontinuous if and only if it is almost density-equicontinuous.

**Proof: (Necessity)** For any  $\varepsilon > 0$ , by the density-equicontinuity of  $(X, d, T)$ , there exists a  $\delta > 0$  such that for any  $x, y \in X$  with  $d(x, y) < \delta$ , the upper density

$$\bar{D}(F_{x,y}) = \bar{D}(\{k \in \mathbb{Z}_+ : d(T^k x, T^k y) > \varepsilon\}) = 0.$$

Due to every point in a minimal t.d.s. is a transitive point, fix a transitive point  $x \in X$  and define its neighborhood  $U = B(x_0, \delta/2)$ .

For any  $u, v \in U$ , the triangle inequality gives  $d(u, v) \leq d(u, x_0) + d(x_0, v) < \delta$ . Then,  $\bar{D}(F_{u,v}) = 0$ , where  $F_{u,v} = \{k \in \mathbb{Z}_+ : d(T^k u, T^k v) > \varepsilon\}$ .

Thus,  $x_0$  is a density-equicontinuous transitive point, so the system is almost density-equicontinuous.

**(Sufficiency)** Let  $\varepsilon > 0$ . Since  $(X, d, T)$  is almost density-equicontinuous, there exist a transitive point  $x_0$  and a  $\delta_1 > 0$  such that for any  $u, v \in B(x_0, \delta_1)$ ,

$$\bar{D}(\{k \in \mathbb{Z}_+ : d(T^k u, T^k v) > \varepsilon\}) = 0.$$

By Lemma 3.1, there exists an  $n \in \mathbb{Z}_+$  such that  $T^n x = u \in U$  and  $T^n y = v \in U$ .

By the continuity of  $T^n$ , for the above  $\delta_1 > 0$ , there exists a  $\delta > 0$  such that  $d(x, y) < \delta$  implies  $d(u, v) = d(T^n x, T^n y) < \delta_1$ . By the iterative property of t.d.s.,  $T^{n+k} x = T^k u$  and  $T^{n+k} y = T^k v$  for all  $k \in \mathbb{Z}_+$ . Then, for  $k < n$ , one can get that  $F_{x,y}^{(1)} = \{k \in \mathbb{Z}_+ : k < n \text{ and } d(T^k x, T^k y) > \varepsilon\}$  is finite, so  $\bar{D}(F_{x,y}^{(1)}) = 0$ . And for  $k \geq n$ , one has  $F_{x,y}^{(2)} = \{k \in \mathbb{Z}_+ : k \geq n \text{ and } d(T^k x, T^k y) > \varepsilon\}$  coincides with  $F_{u,v}$ , so  $\bar{D}(F_{x,y}^{(2)}) = 0$ .

By the subadditivity of upper density,

$$\bar{D}(F_{x,y}) = \bar{D}(F_{x,y}^{(1)} \cup F_{x,y}^{(2)}) \leq \bar{D}(F_{x,y}^{(1)}) + \bar{D}(F_{x,y}^{(2)}) = 0.$$

Thus, for any  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that  $d(x, y) < \delta$  implies  $\bar{D}(F_{x,y}) = 0$ , so the system is density-equicontinuous.

Therefore, in a minimal t.d.s., density-equicontinuity is equivalent to almost density-equicontinuity.

Minimal systems with core setting of the above equivalence have distinct dynamical structures. Glasner studied rigidity in minimal flows, linking uniform rigidity to the diagonal invariant relation and constructing rigid almost automorphic minimal systems ([18]). These results highlight the specific properties of minimal systems, which support the density-based equicontinuity equivalence here.

**Remark 3.1:** If the system is not minimal, the relationship in Theorem 3.3 not always holds, as illustrated by the following example.

**Example 3.2:** There exists a t.d.s. which is density-equicontinuity but not almost density-equicontinuous.

**Proof:** Let  $X$  be a set equipped with the discrete metric  $d(x, y)$ , where  $d(x, y) = 0$  if  $x = y$  and  $d(x, y) = 1$  otherwise. Let  $T : X \rightarrow X$  denotes the identity map, i.e.,  $T(x) = x$  for all  $x \in X$ .

For any  $\varepsilon > 0$ , take  $\delta = 1$ . Under the discrete metric, if  $d(x, y) < \delta$ , then  $x = y$  necessarily holds. For such  $x$  and  $y$ , the orbit separation set  $F = \{k \in \mathbb{Z}_+ : d(T^k x, T^k y) > \varepsilon\}$  is empty, so its upper density  $\bar{D}(F) = 0$ . According to the definition of density- $t$ -equicontinuity, if  $t = 1$ , the condition  $\bar{D}(F) \leq 1 - t$  is equivalent to  $\bar{D}(F) = 0$ . So, the system satisfies the definition of density-equicontinuity.

Almost density-equicontinuous requires the existence of transitive points, meaning a point  $x_0 \in X$  exists with the orbit  $\text{orb}(x_0, T)$  dense in  $X$ . Yet, for the identity map,  $\text{orb}(x, T) = \{x\}$  for all  $x \in X$ , failing to be dense in  $X$ . Thus, there are no transitive points

in  $X$ , meaning that the system cannot satisfy the definition of almost density-equicontinuity.

The above example illustrate that density- $t$ -equicontinuity and almost density-equicontinuity are not equivalent in general t.d.s. This is because density- $t$ -equicontinuity is a global quantitative constraint, while almost density-equicontinuity is a local qualitative constraint. Global constraints cannot be covered by local constraints, and local constraints rely on the existence of transitive points as a premise. This premise does not hold in many general systems, such as the identity map system in Example 3.2. Therefore, in Theorem 3.3, the equivalent relationship can be obtained between the two properties only in minimal t.d.s.

### DENSITY-BASED EQUICONTINUITIES INHIBIT CHAOS

The inhibitory effect of density-based equicontinuity on chaos is first investigated, based on the equivalence results between density- $t$ -equicontinuity and almost density-equicontinuity in minimal systems in section 3. The main conclusions are as follows.

**Theorem 4.1:** If a minimal t.d.s.  $(X, d, T)$  is density-equicontinuous, it cannot be Devaney chaotic.

**Proof:** Assume that the minimal t.d.s.  $(X, d, T)$  is both density-equicontinuous and Devaney chaotic.

By Remark 4.4 (1) in [10], density-equicontinuity implies equicontinuity for minimal systems. Formally, for any  $\varepsilon > 0$ , a  $\delta > 0$  exists such that for any  $x, y \in X : d(x, y) < \delta$  and any  $k \in \mathbb{Z}_+$ ,  $d(T^k x, T^k y) < \varepsilon$  holds.

By Devaney chaos, the system must sensitive dependence on initial conditions, i.e., there exists a  $\gamma > 0$  such that for any  $x \in X$  and any  $\delta > 0$ , there exist a  $y \in B(x, \delta)$  and an  $n \in \mathbb{Z}_+$  satisfying  $d(T^n x, T^n y) > \gamma$ .

For the sensitive constant  $\gamma > 0$  mentioned above, equicontinuity guarantees a  $\delta_0 > 0$  such that for any  $x, y \in X$  with  $d(x, y) < \delta_0$ ,  $d(T^k x, T^k y) < \gamma$  for every  $k \in \mathbb{Z}_+$ . This means that for any  $x \in X$ , any  $y \in B(x, \delta_0)$ ,  $d(T^k x, T^k y) < \gamma$  for all iterations  $k$ , making it impossible to find a  $y \in B(x, \delta_0)$  and an  $n \in \mathbb{Z}_+$  such that  $d(T^n x, T^n y) > \gamma$ .

This directly violates the sensitive dependence requirement, a fundamental criterion for Devaney chaos. Therefore, a density-equicontinuous minimal t.d.s. cannot be Devaney chaotic.

Researchers have studied systems with no Li-Yorke pairs. Blanchard [19] showed that such systems can be minimal if they are transitive. Additionally, the properties of these systems are stable under factor maps. Based on these findings, the following results about density-equicontinuous systems and Li-Yorke pairs can be obtained.

**Theorem 4.2:** If a t.d.s.  $(X, d, T)$  satisfies density-equicontinuity, the upper density of the Li-Yorke scrambled set in  $X \times X$  is zero.

**Proof:** Assume that a set  $LY(f) \subset X \times X$  of Li-Yorke pairs has upper density  $\bar{D}(LY(f)) > 0$ . By the definition of Li-Yorke pairs, for any  $(x, y) \in LY(f)$ ,

$$\liminf_{k \rightarrow \infty} d(T^k x, T^k y) = 0 \quad \text{and} \quad \limsup_{k \rightarrow \infty} d(T^k x, T^k y) = \alpha > 0.$$

Let  $\varepsilon = \alpha/3 > 0$ . Since  $(X, d, T)$  is density-equicontinuous, a  $\delta > 0$  exists such that for all  $x, y \in X$  satisfying  $d(x, y) < \delta$ ,  $F = \{k \in \mathbb{Z}_+ : d(T^k x, T^k y) > \varepsilon\}$  has upper density  $\bar{D}(F) = 0$ .

Take any  $(x, y) \in LY(f)$ . Since  $\limsup_{k \rightarrow \infty} d(T^k x, T^k y) = \alpha$ , there are infinitely many  $k$  satisfying  $d(T^k x, T^k y) > 2\varepsilon$ , where  $2\varepsilon = 2\alpha/3 < \alpha$ . Let  $G = \{k \in \mathbb{Z}_+ : d(T^k x, T^k y) > 2\varepsilon\}$ , by the definition of upper density,  $\bar{D}(G) > 0$ .

However, by density-equicontinuity,  $d(x, y) < \delta$  guarantees  $\bar{D}(F) = 0$ . Note that  $d(T^k x, T^k y) > 2\varepsilon$  implies  $d(T^k x, T^k y) > \varepsilon$ , then  $G \subset F$ . So  $\bar{D}(G) \leq \bar{D}(F) = 0$ , this contradict to  $\bar{D}(G) > 0$ .

Therefore, the upper density of Li-Yorke scrambled set within  $X \times X$  is zero.

**Example 4.1:** Endow  $X = [0, 1] \cup [2, 3]$  with the Euclidean distance  $d(x, y) = |x - y|$ . The continuous map  $T : X \rightarrow X$  is given by

$$T(x) = \begin{cases} \frac{x}{2} & x \in [0, 1]; \\ 3.8x(1-x) & x \in [2, 3]. \end{cases}$$

Take  $x_0 = 0$  and a neighborhood  $U = [0, \delta)$  ( $\delta < 1$ ). For any  $x, y \in U$  with  $d(x, y) < \delta$ , the orbits satisfy  $T^k(x) = \frac{x}{2^k}$  and  $T^k(y) = \frac{y}{2^k}$  for  $k \in \mathbb{Z}_+$ . As  $k \rightarrow \infty$ ,  $d(T^k x, T^k y) = \frac{d(x, y)}{2^k} \rightarrow 0$ . So, for any  $\varepsilon > 0$ , there exists an  $N \in \mathbb{Z}_+$  such that  $d(T^k x, T^k y) < \varepsilon$  for all  $k \geq N$ . The orbit separation set  $F = \{k \in \mathbb{Z}_+ \mid d(T^k x, T^k y) > \varepsilon\}$  satisfies  $F \subseteq [0, N-1]$ , and its upper density  $\bar{D}(F) = 0$ . Thus, the system is density-equicontinuous on the subspace  $[0, 1]$ .

The map  $T|_{[2, 3]}(x) = 3.8x(1-x)$  is an extended Logistic map. For the parameter  $r = 3.8$ , it satisfies the core conditions of Li-Yorke chaos as shown in [20]. The additional

results of Shi[21] on the chaos of Logistic map confirm that the system exhibits Li-Yorke chaos on  $[2,3]$ .

This example illustrates that density-equicontinuity can hold on a specific subspace  $([0,1])$  while the system displays chaotic behavior on another subspace  $([2,3])$ , which reflects the non-uniform stability of dynamical systems.

**Remark 4.1:** (1) By Theorem 3.3, density-equicontinuity is equivalent to almost density-equicontinuity for a minimal t.d.s. Combining with Theorem 4.1, a minimal t.d.s. that is almost density-equicontinuous cannot be Devaney chaotic.

(2) According to reference [22], Devaney chaos implies Li-Yorke chaos. Together with Theorem 4.1 and the equivalence of density-equicontinuity and almost density-equicontinuity in minimal t.d.s., a minimal t.d.s. that is density-equicontinuous or almost density-equicontinuous cannot be Li-Yorke chaotic.

## CONCLUSIONS

This study explores the relationships between density-equicontinuity, mean equicontinuity, density- $t$ -equicontinuity, almost density-equicontinuity, and chaoticity of t.d.s. The results illustrate the logical relationship between various concepts of density-based equicontinuity, and verify the inhibitory effect of density-based equicontinuity in chaotic dynamics, thus improving the relevant theory of dynamic systems. Nevertheless, the current research is limited to the setting of density, and there are several questions that need be resolved.

- **Question 5.1:** For density-based equicontinuity systems, are there chaos inhibition or judgment criteria?
- **Question 5.2:** For density-equicontinuity (or almost density-equicontinuity, or density- $t$ -equicontinuity), are there some internal correlation between product systems and its factor systems? How does these internal correlations affect the chaotic behavior of the product systems?

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