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Cayley Coset Digraphs

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Abstract: In this paper, a new class of digraphs, namely, Cayley coset digraphs Γ(G,H,S) associated with a group G, a subgroup H and a subset S of G is introduced and it is shown that it is vertex transitive. Further the conditions under which this digraph is connected, complete and has loops and parallel edges are obtained.

Keywords: Digraph, Cayley coset digraph, vertex transitivity, connected graph, complete graph, loop, parallel edges.

AMS Subject Classification: 05C20, 05C25, 05C40, 68R10

INTRODUCTION

The study of Cayley graphs has its origin, when Konig [14] posed the problem of finding a graph, associated with a group G, such that the group of automorphisms of the graph Γ is isomorphic to the group G. In 1938, Frucht [10] solved this problem by constructing a digraph Γ associated with a group G and a subset S of G whose vertices are elements of G and whose edges are defined by using the left (or, right) regularizations of the elements of G by the elements of S, which are akin to those used in Cayley's theorem for groups. The graph Γ so obtained is named as the Cayley digraph associated with a group G and its subset S, which is denoted by $\Gamma(G,S)$. This digraph has G as vertex set and the set $E = \{(x,y)/\ y = xs, \text{for some } s \in S\}$. This triggered an active research on Cayley graphs by many Graph Theorists. To mention some of them, Berrizbietia and Guidici [1,2], Dejter and Guidici [6], Imrich and Watkins [12] and Joy Morris [13] have extensively studied various aspects of Cayley graphs. Madhavi et.al., [7,16,17] have studied Cayley graphs associated with some arithmetic functions, namely, Euler Totient function, quadratic residues modulo a prime, divisor function, zero-divisors and nilpotent elements in the ring $(Z_n,\oplus,\odot)$ modulo an integer $n \geq 1$.

There is another class of Cayley digraphs called the Cayley coset digraphs associated with a group G, a subgroup H of G and a subset S of G, which are studied by Cai Hang Li [5], Emanuel Knill [8], Kurcz [15] and Robert Rosiek [20]. Venkat Subbaiah [21] studied the automorphisms of Cayley conjugate coset digraphs.

In this paper, the authors study the conditions under which the Cayley coset digraph is connected and has loops and parallel edges.

The reader is referred to J.A. Bondy and U.S.R. Murty [4], Narsingh Deo [18] and Frank Harary [9] for graph theory, I. N. Herstein [11] and Bhattacharya [3] for group theoretic terminology and notations that are not explained here.

THE CAYLEY COSET DIGRAPH $\Gamma(G,H,S)$ AND ITS BASIC PROPERTIES

Definition 2.1: Let H be a subgroup of a group G and S a subset of G such that $H \cap S \neq \emptyset$. The Cayley coset digraph $\Gamma(G,H,S)$ has the vertex set $V = G/H$, the set of all left cosets of H in G and the edge set E is the set of all ordered pairs (aH, bH) of vertices such that $aHs \cap bH \neq \emptyset$ for some $s \in S$. Here the element s in S is called the **color** of the edge (aH, bH).

Clearly, the Cayley coset digraph $\Gamma(G,H,S)$ is a digraph.

Remark 2.2: In the definition 2.1 above, one can as well take the right cosets of H in G. However, the results in this case will be similar to those with left cosets of H in G.Throughout our study we consider the set G/H of left cosets of H in G and a coset always means a left coset of H in G.

An edge of $\Gamma(G,H,S)$ is defined between left cosets of $\Gamma(G,H,S)$ of H in G. However left coset has different representations, so that it is necessary to show that the edge (aH, bH) is the same for all representations of aH and bH. To see this, let $aH = a_1H$ and $bH = b_1H$, for some $a, a_1, b, b_1 \in G$ and let (aH, bH) be an edge with color $s \in S$. Since (aH, bH) is an edge of $\Gamma(G,H,S)$ with color $s \in S$, we have

$$aHs \cap bH \neq \emptyset \Rightarrow \exists x \in aHs \cap bH \Rightarrow x \in aHs \text{ and } x \in bH.$$

$\Rightarrow x = ah_1s$ and $x = bh_2$, for some $h_1, h_2 \in H$.

Further, $aH = a_1H$ and $h_1 \in H \Rightarrow ah_1 = a_1h_3$, for some $h_3 \in H$ and $bH = b_1H$ and $h_2 \in H \Rightarrow bh_2 = b_1h_4$, for some $h_4 \in H$. So, $x = ah_1s$ and $x = bh_2 = b_1h_4$, for some $h_1, h_2, h_3, h_4 \in H$, so that, $x = ah_1s = a_1h_3s \in a_1Hs$ and $x = bh_2 = b_1h_4 \in b_1H$. That is, $x \in a_1Hs$ and $x \in b_1H \Rightarrow x \in a_1Hs \cap b_1H \Rightarrow a_1Hs \cap b_1H \neq \emptyset$. This shows that (a_1H, b_1H) is also an edge with the same color s as that of (aH, bH). So (aH, bH) and (a_1H, b_1H) represents the same edge with color s.

Definition 2.3: Let G be a group, H a subgroup and S a subset of G. For the vertices aH, bH in $\Gamma(G,H,S)$, if (aH, bH) is an edge of $\Gamma(G,H,S)$ then (aH, bH) is called an **out-edge** at aH and an **in-edge** at bH.

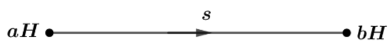

Fig 1: Out-edge at aH and In-edge at bH

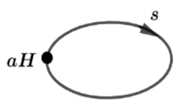

Fig 2: Out-edge and In-edge in the loop (aH, aH)

Theorem 2.4: For each $s \in S$ and $aH, bH \in G/H$, the following are equivalent:

1. (aH, bH) is an out-edge at aH corresponding to the color s.
2. $a^{-1}b \in HsH$.
3. $bH \subseteq aHsH$.

Proof: For $s \in S$ and $aH, bH \in G/H$ let (aH, bH) be an out-edge at aH corresponding to the color s. Then $(aHs) \cap bH \neq \emptyset$. So, there exists $x \in (aHs) \cap bH$ such that $x \in aHs$ and

$x \in bH$, so that $x = ah_1 s$ and $x = bh_2$, for some $h_1, h_2 \in H$. That is, $ah_1 s = bh_2 \Rightarrow h_1 s = a^{-1} b h_2 \Rightarrow h_1 s {h_2}^{-1} = a^{-1} b$. Since H is a subgroup of G, $h_2 \in H$ implies that ${h_2}^{-1} \in H$, so that $a^{-1}b \in HsH$ and this shows that (i) ⇒ (ii).

Let $a^{-1}b \in HsH$. Then $b \in aHsH \Rightarrow bH \subseteq aHsHH \Rightarrow bH \subseteq aHsH$, since $HH = H$. That is, (ii) ⇒ (iii).

Let $bH \subseteq aHsH$. Then $bh_1 = ah_2 s h_3$, for some $h_1, h_2, h_3 \in H$.

$\Rightarrow bh_1 {h_3}^{-1} = ah_2 s \Rightarrow bh_4 = ah_2 s$, where $h_4 = h_1 {h_3}^{-1}$.

Since $bh_4 \in bH$ and $ah_2 s \in aHs$, it follows that $(aHs) \cap bH \neq \emptyset$. So, (aH, bH) is an out-edge from aH to bH with colour s and this shows that (iii) ⇒ (i). ■

Remark 2.5: In view of the above theorem, (aH, bH) is an out-edge at aH with respect to the color s if $a^{-1}b \in HsH$. So the set $E = \{(aH, bH) \,/\, aH, bH \in G/H$ and $a^{-1}b \in HsH\}$ is the edge set of the graph $\Gamma(G, H, S)$.

The analogous theorem for an in-edge at aH with respect to the color s is as follows and its proof is analogous to that of Theorem 2.4.

Theorem 2.6: For each $s \in S$ and $aH, bH \in G/H$, the following are equivalent:

1. (aH, bH) is an in-edge at aH corresponding to the color s.
2. $a^{-1}b \subseteq Hs^{-1}H$.
3. $bH \subseteq aHs^{-1}H$.

Example 2.7: Let $G = S_3$ be the symmetric group on the set $\{1,2,3\}$. That is, $S_3 = \{e, \alpha, \beta, \gamma, \delta, \theta\}$, where e is the identity map, $\alpha = (1\,2\,3)$, $\beta = (1\,3\,2)$, $\gamma = (2\,3)$, $\delta = (1\,3)$ and $\theta = (1\,2)$. For the subgroup $H = \{e, \gamma\}$ and the subset $S = \{\alpha, \delta\}$ of S_3, the digraph $\Gamma(S_3, H, S)$ has the vertex set $G/H = S_3/H = \{H, \alpha H, \beta H\}$. Here $H = \{e, \gamma\}$, $\alpha H = \{\alpha, \theta\}$, $\beta H = \{\beta, \delta\}$. We also have, $H\alpha = \{\alpha, \delta\}$, $(\alpha H)\alpha = \{\beta, \gamma\}$, $(\beta H)\alpha = \{e, \theta\}$, $H\delta = \{\delta, \alpha\}$, $(\alpha H)\delta = \{\gamma, \beta\}$ and $(\beta H)\delta = \{\theta, e\}$.

<u>Table 2.1</u>: Table giving the edges in the Cayley coset digraph $\Gamma(S_3, H, S)$

	$H = \{e,\gamma\}$	$\alpha H = \{\alpha,\theta\}$	$\beta H = \{\beta,\delta\}$
$H\alpha = \{\alpha,\delta\}$	$\emptyset$	$\{\alpha\}$	$\{\delta\}$
$(\alpha H)\alpha = \{\beta,\gamma\}$	$\{\gamma\}$	$\emptyset$	$\{\beta\}$
$(\beta H)\alpha = \{e,\theta\}$	$\{e\}$	$\{\theta\}$	$\emptyset$
$H\delta = \{\delta,\alpha\}$	$\emptyset$	$\{\alpha\}$	$\{\delta\}$
$(\alpha H)\delta = \{\gamma,\beta\}$	$\{\gamma\}$	$\emptyset$	$\{\beta\}$
$(\beta H)\delta = \{\theta,e\}$	$\{e\}$	$\{\theta\}$	$\emptyset$

From the above table one can observe that $H \cap H\alpha = \emptyset$, so that there is no edge from H to H with respect to color $'\alpha'$. Also $(\beta H)\alpha \cap \alpha H = \{\theta\} \neq \emptyset$, so there is an edge from βH to αH with respect to color $'\beta'$. Similarly, $\alpha Hs \cap \beta H = \{\beta\} \neq \emptyset$, so there is an edge from αH to βH. In this way, one can find the edges from the vertices of $\Gamma(S_3, H, S)$, and the digraph $\Gamma(S_3, H, S)$ is as follows:

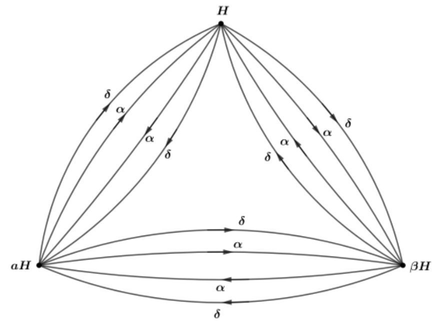

Figure 2.1: The digraph $\Gamma(S_3, H, S)$

Remark 2.8: Observe that S is a generating subset of S_3 and $\Gamma(S_3, H, S)$ is a connected graph.

Example 2.9: Let us take $G = S_4$ and $H = \{e, (2\ 3)\}$ and $S = \{\alpha\}$, where $\alpha = (2\ 3\ 4)$. The left cosets of H in G are:

$A_0 = H = \{e, (2\ 3)\}$, $A_1 = (3\ 4)\ H = \{(3\ 4), (2\ 4\ 3)\}$,

$A_2 = (2\ 3\ 4)H = \{(2\ 3\ 4), (2\ 4)\}$, $A_3 = (1\ 2)H = \{(1\ 2), (1\ 2\ 3)\}$,

$A_4 = (1\ 2)(3\ 4)H = \{(1\ 2)(3\ 4), (1\ 2\ 4\ 3)\}$,

$A_5 = (1\ 2\ 3\ 4)H = \{(1\ 2\ 3\ 4), (1\ 2\ 4)\}$,

$A_6 = (1\ 3\ 2)H = \{(1\ 3\ 2), (1\ 3),$

$A_7 = (1\ 3\ 4\ 2\ )H = \{(1\ 3\ 4\ 2), (1\ 3)(2\ 4)\}$,

$A_8 = (1\ 3\ 4)H = \{(1\ 3\ 4), (1\ 3\ 2\ 4)\}$,

$A_9 = (1\ 4\ 3\ 2)H = \{(1\ 4\ 3\ 2), (1\ 4\ 3)\}$,

$A_{10} = (1\ 4\ 2)H = \{(1\ 4\ 2), (1\ 4\ 2\ 3)\}$,

$A_{11} = (1\ 4)H = \{(1\ 4), (1\ 4)(2\ 3)\}$.

Moreover,

$A_0\alpha = \{(2\ 3\ 4), (3\ 4)\}$, $A_1\alpha = \{(2\ 4), e\}$,

$A_2\alpha = \{(2\ 4\ 3), (2\ 3)\}$, $A_3\alpha = \{(1\ 2\ 3\ 4), (1\ 2)(3\ 4)\}$,

$A_4\alpha = \{(1\ 2\ 4), (1\ 2)\}$, $A_5\alpha = \{(1\ 2\ 4\ 3), (1\ 2\ 3)\}$,

$A_6\alpha = \{(1\ 3\ 4), (1\ 3\ 4\ 2)\}$, $A_7\alpha = \{(1\ 3\ 2\ 4), (1\ 3\ 2)\}$,

$A_8\alpha = \{(1\ 3)(2\ 4), (1\ 3)\}$, $A_9\alpha = \{(1\ 4), (4\ 2)\}$,

$A_{10}\alpha = \{(1\ 4)(2\ 3), (1\ 4\ 3\ 2)\}$, $A_{11}\alpha = \{(1\ 4\ 2\ 3), (1\ 4\ 3)\}$.

Using these, one can see that the digraph $\Gamma(S_4, H, S)$ is given as below:

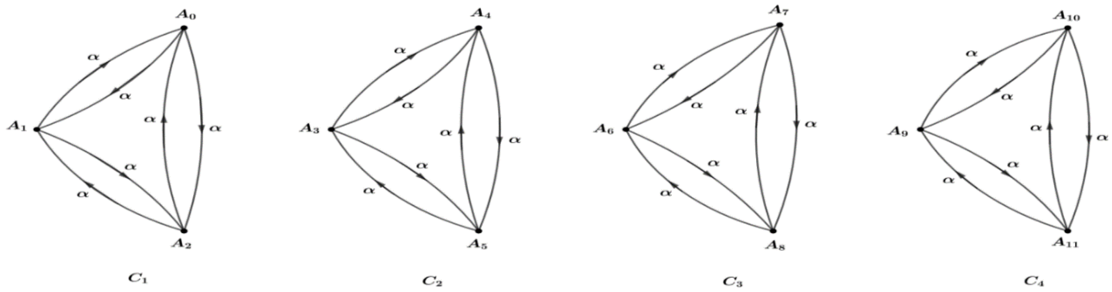

Figure 2.2: The Cayley coset digraph $\Gamma(S_4, H, S)$

Remark 2.10: Observe that S is **not** a generating set of S_4 and the Cayley coset digraph $\Gamma(S_4, H, S)$ is disconnected and contains four components C_1, C_2, C_3 and C_4.

Definition 2.11: The digraph $\mathcal{D}$ is said to be **vertex transitive**, if for any two vertices u and v in $\mathcal{D}$, there is an automorphism θ of $\mathcal{D}$ such that $\theta(a) = b$.

Theorem 2.12: The digraph $\Gamma(G, H, S)$ is vertex transitive.

Proof: The digraph $\Gamma(G, H, S)$ is vertex transitive, if for every pair of vertices aH and bH in $\Gamma(G, H, S)$, there is an automorphism θ of $\Gamma(G, H, S)$ such that $\theta(aH) = bH$.

Let $aH, bH \in G/H$. Then $a, b \in G$, so that $ba^{-1} \in G$. Define the mapping $\theta: \Gamma(G, H, S) \to \Gamma(G, H, S)$ by

$$\theta(gH) = (ba^{-1})gH, \forall gH \in G/H.$$

To show that θ is well-defined (as θ is defined on a left coset, which has many representations), let $gH = kH$. Then $k = ke \in kH = gH \Rightarrow k = g$, for some $h \in H$. Now

$$h \in H \Rightarrow \ \theta(kH) = (ba^{-1})kH = (ba^{-1})ghH = (ba^{-1})gH \text{ (since } hH = H) = \theta(gH)$$

$$\Rightarrow \ \theta(kH) = \theta(gH), \text{ so that } \theta \text{ is well-defined.}$$

One-to-one-ness of $\boldsymbol{\theta}$: For $gH, kH \in G/H$, $\theta(gH) = \theta(kH)$

$\Rightarrow \ (ba^{-1})gH = (ba^{-1})kH \Rightarrow \ (ba^{-1})gh_1 = (ba^{-1})kh_2$, for some $h_1, h_2 \in H$

$\Rightarrow \ gh_1 = kh_2$, by the left cancellation law in the group $(G, \cdot)$

$\Rightarrow \ (gh_1)H = (kh_2)H \Rightarrow \ gH = kH$, since $h_1H = H$ and $h_2H = H$.

So θ is **one-to-one**.

Onto-ness of $\boldsymbol{\theta}$: Let $kH \in G/H$. Then $k \in G$ and $g = ab^{-1}k \in G$, so that $gH \in G/H$ and

$\theta(gH) = (ba^{-1})(gH) = (ba^{-1})(ab^{-1}k)H = kH.$

This shows that gH is a pre-image of kH and thus θ is **onto**.

To show that θ is an automorphism of $\Gamma(G, H, S)$, let (gH, kH) be an edge of $\Gamma(G, H, S)$with color s. Then $g^{-1}k \in HsH$. Now

$$\theta(gH) = (ba^{-1})gH \text{ and } \theta(kH) = (ba^{-1})(kH).$$

Also $(ba^{-1}g)^{-1}(ba^{-1}k) = g^{-1}ab^{-1}ba^{-1}k = g^{-1}k \in HsH$.

So $(\theta(gH), \theta(kH))$ is also an edge with color s.

Next suppose that $\big(\theta(gH), \theta(kH)\big)$ is an edge with color s

$\Rightarrow$ $\big((ba^{-1})gH,\ (ba^{-1})kH\big)$ is an edge with color s

$\Rightarrow$ $(ba^{-1}g)^{-1}(ba^{-1})k \in HsH \Rightarrow\ g^{-1}k \in HsH$

$\Rightarrow$ (gH, kH) is an edge with color s.

These show that θ is an automorphism of the digraph $\Gamma(G,H,S)$.

Finally, $\theta(aH) = (ba^{-1})aH = bH$, which shows that θ maps aH to bH. Thus $\Gamma(G,H,S)$ is vertex transitive. ■

Theorem 2.13: The graph $\Gamma(G,H,S)$ is a trivial graph if, and only if, $S = \emptyset$.

Proof: Suppose that $S = \emptyset$. If possible, assume that $\Gamma(G,H,S)$ is not a trivial graph. Then the edge set E of $\Gamma(G,H,S)$ is non-empty. So there exist an edge $(xh, yH) \in E$, with color $s \in S$, so that $x^{-1}y \in HsH$. This shows that $S \neq \emptyset$, which is a contradiction to the assumption $S = \emptyset$. So the edge set E must be empty and the graph $\Gamma(G,H,S)$ is trivial.

On the other hand, assume that $\Gamma(G,H,S)$ is a trivial graph. Then the edge set $E = \emptyset$. Suppose that $S \neq \emptyset$. Then there exists $s \in S$. Now $S \subseteq G$ implies that $s \in G$. The edges eH and sH are such that $e^{-1}s = es = ese \in HsH$ (since $e \in H$), so that (eH, sH) is an edge with color s and thus $E \neq \emptyset$. This is a contradiction to the fact that $E = \emptyset$. So, the assumption $S \neq \emptyset$ is wrong and thus $S = \emptyset$. ■

Theorem 2.14: The graph $\Gamma(G,H,S)$ is a complete graph if, and only if, $G = HSH$.

Proof: Suppose that $\Gamma(G,H,S)$ is a complete graph. Then for every $x \in G$ and for every $xH \in G/H$, (eH, xH) is an edge of $\Gamma(G,H,S)$

$\Rightarrow$ $e^{-1}x \in HsH$, for some $s \in S \Rightarrow\ x \in HSH \Rightarrow\ G \subseteq HSH$.

Trivially $HSH \subseteq G$, so that $G = HSH$.

Conversely, let $G = HSH$ and let $gH, kH \in G/H$.

Then $g, k \in G$, so that $g^{-1}k \in G = HSH$

$\Rightarrow$ $g^{-1}k = h_1 s h_2$, for some $h_1, h_2 \in H$ and $s \in S$.

$\Rightarrow$ $g^{-1}k \in HsH \Rightarrow\ (gH, kH)$ is an edge with color $s \in S$.

So, there is an edge between every pair of vertices in $\Gamma(G,H,S)$, so that the graph $\Gamma(G,H,S)$ is a complete graph. ■

Theorem 2.15: The graph $\Gamma(G,H,S)$ is a connected graph if, and only if, $G = \langle HSH \rangle$, the subgroup generated by the subset HSH.

Proof: The digraph $\Gamma(G,H,S)$ is connected if, and only if, there is a directed path from gH and kH, for every pair of vertices gH and kH in $\Gamma(G,H,S)$.

Let E be the edge of the digraph $\Gamma(G,H,S)$. Assume that $\Gamma(G,H,S)$ is a connected graph. Then there exists a directed path from $H = (eH)$ to gH, for every vertex gH in $\Gamma(G,H,S)$, say,

$$H \longrightarrow g_1H \longrightarrow g_2H \longrightarrow \cdots \longrightarrow g_nH \longrightarrow gH,$$

for some vertices $g_1H, g_2H, \ldots, g_nH$ in $\Gamma(G,H,S)$.

Now $(H, g_1H) \in E \Rightarrow e^{-1}g_1 \in Hs_1H$, for some $s_1 \in S$

$\Rightarrow g_1 \in Hs_1H$, for some $s_1 \in S$

$\Rightarrow g_1 = h_1s_1h_1'$, for some $h_1, h_1' \in H$ and $s_1 \in S$.

Also, $(g_1H, g_2H) \in E \Rightarrow g_1^{-1}g_2 \in Hs_2H$, for some $s_2 \in S$

$\Rightarrow g_1^{-1}g_2 = h_2s_2h_2'$ for some $h_2, h_2' \in H$ and $s_2 \in S$

$\Rightarrow g_2 = g_1(h_2s_2h_2')$

$\Rightarrow g_2 = (h_1s_1h_1')(h_2s_2h_2')$, for some $h_1, h_1', h_2, h_2' \in H$ and $s_1, s_2 \in S$.

Proceeding in this way, after $(n+1)$ steps, we have

$$g = (h_1s_1h_1')(h_2s_2h_2') \dots (h_{n+1}s_{n+1}h_{n+1}') \in \langle HSH \rangle,$$

since $h_is_ih_i' \in HSH$, for $1 \leq i \leq n+1$.

Now $gH \in G/H \Rightarrow g \in G \Rightarrow \forall\, g \in G,\ g \in \langle HSH \rangle \Rightarrow G \subseteq \langle HSH \rangle$.

Trivially $\langle HSH \rangle \subseteq G$, so that $G = \langle HSH \rangle$.

Conversely, assume that $G = \langle HSH \rangle$. Let $gH, kH \in G/H$, then $g, k \in G$. So there exists an element $t \in G$ such that $k = gt$. Now $t \in G$ and $G = \langle HSH \rangle$ implies that

$$t = t_1t_2 \dots t_n, \text{ for some } t_1, t_2, \dots, t_n \in HSH.$$

Let $g_1 = et_1 = t_1,\ g_2 = g_1t_2,\ g_3 = g_2t_3,\ \dots,\ g_n = g_{n-1}t_n$.

For $i = 1, 2, 3, \dots, n$, $g_{i-1}^{-1}g_i = t_i \in HSH \Rightarrow (g_{i-1}H, g_iH)$ is an edge of $\Gamma(G, H, S)$, so that

$$eH = H \longrightarrow g_1H \longrightarrow g_2H \longrightarrow \cdots \longrightarrow g_nH$$

is a directed path from H to g_nH.

Consider the vertex set $\{gH, gg_1H, gg_2H, \dots, gg_nH\}$. In fact, this is the set of images of $H, g_1H, g_2H, \dots, g_nH$ under the automorphism θ of $\Gamma(G, H, S)$ given by

$$\theta(xH) = (ge^{-1})xH = gxH \text{ (see Theorem 2.12).}$$

For the vertices $gg_{i-1}H, gg_iH$, we have $(gg_i)^{-1}(gg_{i-1}) = g_{i-1}^{-1}g^{-1}gg_i = g_{i-1}^{-1}g_i \in Hs$, for some $s \in S$, (since $(g_{i-1}H, g_iH)$ is an edge of $\Gamma(G, H, S)$), so that

$$gH \longrightarrow gg_1H \longrightarrow gg_2H \longrightarrow \cdots \longrightarrow gg_nH$$

is a directed path from gH to gg_nH.

But $g_n = g_{n-1}t_n = g_{n-2}t_{n-1}t_n = \cdots = t_1t_2 \dots t_n = t$, so that $gg_n = gt = k$.

Thus $gH \longrightarrow gg_1H \longrightarrow gg_2H \longrightarrow \cdots \longrightarrow gg_{n-1}H \longrightarrow kH$ is a directed path from gH to kH, which shows that $\Gamma(G, H, S)$ is a connected digraph. ■

LOOPS

Definition 3.1: An edge with identical ends, say v is called a **loop.** It is treated as an out-edge as well as in-edge at v.

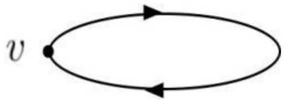

Figure 3.1: The loop (v, v)

Theorem 3.2: Let H be a subgroup of a group G and S a subset of G. Then the Cayley coset digraph $\Gamma(G, H, S)$ has loops if, and only if, $H \cap S \neq \emptyset$.

Proof: Let H be a subgroup of a group G and S a subset of G. Suppose that $H \cap S \neq \emptyset$. Then there exist an element $t \in H \cap S$, such that $t \in H$ and $t \in S$. Since $t \in H$ and H is a subgroup of G, $Ht = H$ and thus

$$HtH = HH = H. \quad \text{------------------------------------} \ (1)$$

Now H being a subgroup of G, $e \in H$, the identity element of G, implies that $e \in HtH$, by (1). So, $e^{-1}e \in HtH$ and thus by Theorem 2.4, (He, He) is an edge with color t, which is a loop in $\Gamma(G, H, S)$.

Conversely, assume that the Cayley coset digraph $\Gamma(G, H, S)$ has a loop at a vertex aH. Then by the Theorem 2.4, (aH, aH) is an edge for some color $s \in S \Rightarrow a^{-1}a \in HsH \Rightarrow e \in HsH$. This gives, for some $h_1, h_2 \in H$,

$$e = h_1 s h_2 \Rightarrow {h_1}^{-1} e {h_2}^{-1} = s \Rightarrow {h_1}^{-1} {h_2}^{-1} = s \Rightarrow (h_2 h_1)^{-1} = s$$

$\Rightarrow s = (h_2 h_1)^{-1} \in H$, since H is a subgroup G.

That is, $s \in S$ and $s \in H \Rightarrow s \in H \cap S$ and thus $H \cap S \neq \emptyset$. ■

The contrapositive result of the above theorem is as follows:

Theorem 3.3: Let H be a subgroup of a group G and S a subset of G. Then the Cayley coset digraph $\Gamma(G, H, S)$ has **no loops** if, and only if, $H \cap S = \emptyset$.

Example 3.4: Let $G = S_3$ be the symmetric group on the set $\{1,2,3\}$. That is, $S_3 = \{e, \alpha, \beta, \gamma, \delta, \theta\}$, where $\alpha = (1\ 2\ 3), \beta = (1\ 3\ 2), \gamma = (2\ 3), \delta = (1\ 3)$ and $\theta = (1\ 2)$. For the subgroup $H = \{e, \gamma\}$ and the subset $S = \{\alpha, \beta, \gamma\}$ of G. Here $\alpha^{-1} = \beta, \beta^{-1} = \alpha$ and $\gamma^{-1} = \gamma$. This graph has the vertex set $G/H = S_3/H = \{H, \alpha H, \beta H\}$, where $H = \{e, \gamma\}, \alpha H = \{\alpha, \theta\}, \beta H = \{\beta, \delta\}$. We also have, $H\alpha = \{\alpha, \delta\}$, $(\alpha H)\alpha = \{\beta, \gamma\}$, $(\beta H)\alpha = \{e, \theta\}$, $H\beta = \{\beta, \theta\}$, $(\alpha H)\beta = \{e, \delta\}$, $(\beta H)\beta = \{\alpha, \gamma\}$, $H\gamma = \{\gamma, e\}$, $(\alpha H)\gamma = \{\theta, \alpha\}$, and $(\beta H)\gamma = \{\delta, \beta\}$.

Table 3.1: The Table giving edges in the digraph $\Gamma(S_3, H, S)$

$\cap$	$H = \{e, \gamma\}$	$\alpha H = \{\alpha\ \theta\}$	$\beta H = \{\beta, \delta\}$
$H\alpha = \{\alpha, \delta\}$	$\emptyset$	$\{\alpha\}$	$\{\delta\}$
$(\alpha H)\alpha = \{\beta, \gamma\}$	$\{\gamma\}$	$\emptyset$	$\{\beta\}$
$(\beta H)\alpha = \{e, \theta\}$	$\{e\}$	$\{\theta\}$	$\emptyset$
$H\beta = \{\beta, \theta\}$	$\emptyset$	$\{\theta\}$	$\emptyset$
$(\alpha H)\beta = \{e, \delta\}$	$\{e\}$	$\emptyset$	$\{\delta\}$
$(\beta H)\beta = \{\alpha, \gamma\}$	$\{\gamma\}$	$\{\alpha\}$	$\emptyset$
$H\gamma = \{\gamma, e\}$	$\{e, \gamma\}$	$\emptyset$	$\emptyset$
$(\alpha H)\gamma = \{\theta, \alpha\}$	$\emptyset$	$\{\alpha, \theta\}$	$\emptyset$
$(\beta H)\gamma = \{\delta, \beta\}$	$\emptyset$	$\emptyset$	$\{\beta, \delta\}$

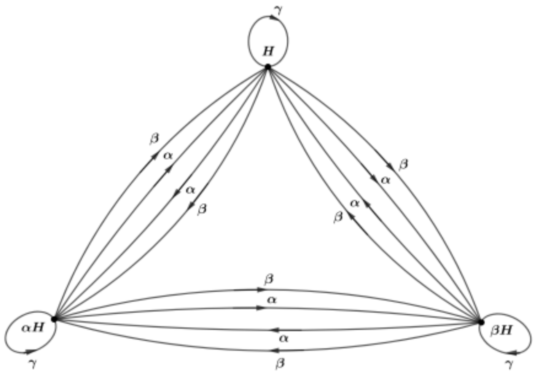

Figure 3.2: The digraph $\Gamma(S_3, H, S)$

Remark 3.5: Here $H \cap S = \{\gamma\} \neq \emptyset$ and the digraph $\Gamma(S_3, H, S)$ **has loops** at the vertices of $H, \alpha H$ and βH. However, in the digraph $\Gamma(S_3, H, S)$ of Example 2.7, where $H = \{e, \gamma\}$ and $S = \{\alpha, \delta\}$, one can observe that $H \cap S = \emptyset$ and $\Gamma(S_3, H, S)$ **has no loops**.

PARALLEL EDGES

Definition 4.1: Let $\Gamma(G, H, S)$ be a Cayley coset digraph with vertex set $G/H = \{gH/g \in G\}$. Let aH, bH be any two vertices in $\Gamma(G, H, S)$. For $s, t \in S,\ s \neq t$ if $(aH, bH)_s$ and $(aH, bH)_t$ are out-edges, then these edges are called parallel edges at aH.

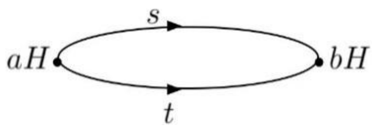

Fig 4.1: Parallel Edges

Definition 4.2: Let $\Gamma(G, H, S)$ be a Cayley coset digraph with vertex set $G/H = \{gH/g \in G\}$. The out-edge (aH, bH) with respect to color $s \in S$ is denoted by $(aH, bH)_s$ and the set of all out-edges at the vertex aH with color s is denoted by E_s. That is, for the vertex aH,

$$E_s = \{(aH, bH)_s / \ bH \in G/H\}.$$

The following theorem gives a necessary and sufficient condition for a pair of vertices in the digraph $\Gamma(G, H, S)$ to have parallel edges and determine the number of parallel edges.

Theorem 4.3: For any two vertices $aH, bH \in \Gamma(G, H, S)$, the edges $(aH, bH)_s$ and $(aH, bH)_t$ are parallel edges if, and only if, $t \in HsH \cap S$.

Proof: For $s \in S$ and $t \in S$, let $(aH, bH) \in E_s \Rightarrow aHs \cap bH \neq \emptyset$ and $(aH, bH) \in E_t \Rightarrow aHt \cap bH \neq \emptyset$. Then by the Theorem 2.4,

$$a^{-1}b \in HsH \text{ and } a^{-1}b \in HtH$$

$\Rightarrow$ for some $h_1, h_2, h_3, h_4 \in H$, $a^{-1}b = h_1 s h_2$ and $a^{-1}b = h_3 t h_4$,

$\Rightarrow h_1 s h_2 = h_3 t h_4 \Rightarrow h_3^{-1} h_1 s h_2 h_4^{-1} = t$

$\Rightarrow t = h_5 s h_6$, where $h_5 = h_3^{-1} h_1 \in H$ and $h_6 = h_2 h_4^{-1} \in H$.

That is, $a^{-1}b = t = h_5sh_6 \in HsH$, since $h_5, h_6 \in H$. So, $t \in HsH \cap S$.

Conversely, suppose that (aH, bH) is an edge with color s. Let $t \in HsH \cap S$.

Then $t \in HsH \cap S \Rightarrow t \in S$ and $t \in HsH$

$\Rightarrow t = h_1sh_2$, for some $h_1, h_2 \in H$.

$\Rightarrow s = h_1^{-1}th_2^{-1}$. ---------- (1)

Suppose $(aH, bH) \in E_s$. Then, by the Theorem 2.4, $a^{-1}b \in HsH$,

$\Rightarrow a^{-1}b \in Hh_1^{-1}th_2^{-1}H$, by (1).

Since H is a subgroup of G, $h_1, h_2 \in H$ implies that $h_1^{-1}, h_2^{-1} \in H$, so that $a^{-1}b \in HtH$.

Again by the Theorem 2.4, $a^{-1}b \in HtH \Rightarrow (aH, bH) \in E_t$.

Therefore, (aH, bH) is also an edge with color t. This shows that the pair of vertices aH and bH have parallel edges with color s and t. ■

Corollary 4.4: For the vertices $aH, bH \in \Gamma(G, H, S)$, the cardinality of the parallel edges from aH to bH is $|S|$.

Proof: By the above Theorem 4.3, $(aH, bH)_s$ and $(aH, bH)_t$ are parallel edges with respect to color s and t from aH to bH if and only if $t \in HsH$. However $t \in S$. So apparently, there are $|HsH \cap S|$ number of parallel edges from aH to bH. Since $e \in H$ for any $s \in S$, we have

$$s = ese \in HsH \Rightarrow s \in HsH \Rightarrow S \subseteq HsH \Rightarrow HsH \cap S = S.$$

Thus, the number of parallel out-edges from aH to bH is $|HsH \cap S| = |S|$. ■

Example 4.5: Let $G = S_3$ be the symmetric group on the set $\{1,2,3\}$. That is, $S_3 = \{e, \alpha, \beta, \gamma, \delta, \theta\}$, where $\alpha = (1\ 2\ 3), \beta = (1\ 3\ 2), \gamma = (2\ 3), \delta = (1\ 3)$ and $\theta = (1\ 2)$. For the subgroup $H = \{e, \theta\}$ and the subset $S = \{\gamma, \delta\}$ of G, the digraph $\Gamma(G, H, S)$ has the vertex set $G/H = S_3/H = \{\theta H, \delta H, \gamma H\}$, where $\theta H = \{\theta, e\}, \delta H = \{\delta, \alpha\}, \gamma H = \{\gamma, \beta\}$. We also have, $H\gamma = \{\gamma, \alpha\}$, $(\delta H)\gamma = \{\beta, \theta\}$, $(\gamma H)\gamma = \{\delta, e\}$, $H\delta = \{\delta, \beta\}$, $(\delta H)\delta = \{e, \gamma\}$ and $(\gamma H)\delta = \{\alpha, \theta\}$.

Table 4.1: The Table giving edges in the the digraph $\Gamma(S_3, H, S)$

$\cap$	$\theta H = \{\theta, e\}$	$\delta H = \{\delta, \alpha\}$	$\gamma H = \{\gamma, \beta\}$
$H\gamma = \{\gamma, \alpha\}$	$\emptyset$	$\{\alpha\}$	$\{\gamma\}$
$(\delta H)\gamma = \{\beta, \theta\}$	$\{\theta\}$	$\emptyset$	$\{\beta\}$
$(\gamma H)\gamma = \{\delta, e\}$	$\{e\}$	$\{\delta\}$	$\emptyset$
$H\delta = \{\delta, \beta\}$	$\emptyset$	$\{\delta\}$	$\{\beta\}$
$(\delta H)\delta = \{e, \gamma\}$	$\{e\}$	$\emptyset$	$\{\gamma\}$
$(\gamma H)\delta = \{\alpha, \theta\}$	$\{\theta\}$	$\{\alpha\}$	$\emptyset$

The Cayley coset digraph $\Gamma(S_3, H, S)$ is as follows

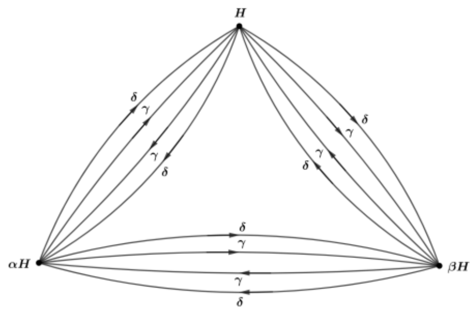

Figure 4.2: The digraph $\Gamma(S_3, H, S)$

The pairs of edges $(H,\alpha H)_\gamma$ and $(H,\alpha H)_\delta$; $(\alpha H,H)_\gamma$ and $(\alpha H,H)_\delta$; $(H,\beta H)_\gamma$ and $(H,\beta H)_\delta$; $(\beta H,H)_\gamma$ and $(\beta H,H)_\delta$; $(\alpha H,\beta H)_\gamma$ and $(\alpha H,\beta H)_\delta$; $(\beta H,\alpha H)_\gamma$ and $(\beta H,\alpha H)_\delta$ are parallel edges in the digraph $\Gamma(S_3,H,S)$.

CONCLUSION

The enumeration of the degree of a vertex and the regularity property of the Cayley conjugate digraph $\Gamma(G,H,S)$ are studied separately.

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