ISSN: 2634-9221
European Journal of Applied Sciences; Vol. 14 No. 02 (2026) (381-406)
https://doi.org/10.14738/aivp.1402.20162

On the "A Priori" Construction of Coordinates in General Relativity

F. F. van Kampen

Abstract: We construct by physical means the Schwarzschild coordinates for a static spherical-symmetric body, the canonical Weyl coordinates for a static axial-symmetric body and the Weyl-Lewis-Papapetrou coordinates for a stationary axial- symmetric body. Because these coordinates are constructed before the equations of GR are solved we speak of the " a priori" construction. The symbols representing these coordinates in this way have a physical meaning. In the usual treatment the symbols representing the coordinates can have an alienating effect because only after the equations of GR are solved, thus "a posteriori", their physical meaning can be unveiled. For this reason, the " a priori" construction contributes to the understanding of the theory. The definition of the second as the unit of time, the phenomena of gravitational time-delay and gravitational red-shift, some peculiarities of the Kerr solution and the occurrences of co-ordinate-singularities are treated as illustrations.

INTRODUCTION

In his book General Relativity and Cosmology G.C. McVittie stated:

"It will be assumed that each event … requires four numbers … to specify it, and these numbers will be called the coordinates of the event. The coordinates of all events under examination by the investigator are presumed to be assigned in the same way, but the particular method employed is not specified a priori. (...) This vagueness in the procedure of assigning coordinates has been criticized by certain applied mathematicians, who maintain that a satisfactory theory must begin by describing the observational methods – actual or conceptual – through which the coordinates are to be set up." [1]

This statement sets the stage for this paper. We construct coordinates by an operational procedure using rigid rods and pulses of light. Following H. Weyl [2] we use the term "construction of the coordinates" although he actually did not give any operational procedure.

We start by assembling in empty space the body we want to investigate by bringing in small masses from large distances. We then describe the observational method to assign to each point outside the body four numbers called its coordinates. To that end we position a small cabin at rest at such a point. The observational method is chosen to be a generalization of the method used in Special Relativity (SR).

Because this construction of the coordinates is based on the behavior of rigid rods and pulses of light we know the experimental values of the metrical functions for each point. The mathematical form of the metrical functions with respect to the constructed coordinates follows from solving the vacuum-equations of GR.

THE METRICAL FUNCTIONS

The expression $ds^2 = \Sigma g_{\mu\nu} dx_\mu dx_\nu$ summed over μ = 1,2,3,4 and ν = 1,2,3,4 for two neighboring point-events P_1 (x_1, x_2, x_3, x_4) and P_2 ($x_1 + dx_1, x_2 + dx_2, x_3 + dx_3, x_4 + dx_4$) does not depend on the coordinates chosen if the functions gμν form the components of the so called metrical tensor. We use the index 4 for the time-coordinate and the indices 1, 2 and 3 for the spatial coordinates. Because we can make the metrical functions $g_{\mu\nu} = g_{\nu\mu}$ by convention, there are for as yet 10 unknown metrical functions.

In the absence of gravity, we can construct Lorentzian coordinates (T, x, y, z) such that for all points we have $ds^2 = 1\, dT^2 - 1/c^2\, (dx^2 + dy^2 + dz^2)$ in which the symbol c denotes the velocity of light in vacuum. From this we see that ds^2 forms a generalization of the Pythagorean theorem of Euclidean Geometry. With respect to these coordinates the metrical functions in Euclidean Geometry are the constants $g_{44} = 1$ and $g_{11} = g_{22} = g_{33} = -1/c^2$, while all other $g_{\mu\nu} = 0$.

In a gravitational field such Lorentzian coordinates exist only for infinitesimal small regions in space and time. We can think of a small cabin free-falling in the gravitational field. In such a cabin we can construct in good approximation Lorentzian coordinates as in SR. We will call these coordinates the cabin-coordinates. In "The Meaning of Relativity" measurements of space-intervals and time-intervals with respect to the cabin coordinates are called the "naturally measured lengths and times" [3].

We can imagine such a cabin reaching at some instant its apex in a point P where it is momentarily at rest with respect to the fixed stars. For a short time-interval around that instant we can relate the cabin-coordinates to the coordinates we are going to construct. To that end we make use of the expressions for ds^2. This can be motivated by the principle of equivalence as discussed by A. Einstein [4] and as further explained by A. Pais [5].

A STATIC SPHERICAL-SYMMETRIC BODY

Assemblage of the Body

We can construct in Lorentzian space-time a spherical surface with the point C as center. To assemble the massive body we compartment the empty sphere in a spherical symmetric way. We can also construct at a near infinite distance from C a second spherical surface we call the Observatory. Then we fill the compartments by bringing in from beyond the Observatory small masses in a spherical symmetric manner. This is a dynamic process in which material tensions in the body in assemblage are generated while also a gravitational field outside the body is building up. These will however not disturb the spherical symmetry. Due to the building up of the gravitational field the initial Lorentzian coordinates will lose their initial meaning. This assemblage is, in view of the large astronomical distances, of course meant to be conceptual rather than actual.

Construction of the Angular Coordinates φ and ψ

Next we construct on the Observatory in the usual way the angular coordinates φ and ψ. We and let φ run from 0 for some point on the equator of the Observatory to 2π and let ψ run from 0 at the north-pole of the Observatory to π at its south-pole.

We divide the large circles through the poles of the Observatory in equal parts by rigid rods of 1 meter and count the number N which make up the circle. Than a rod of 1 meter corresponds to $d\psi(1 \text{ meter}) = 2\pi/N$. The number N is called the circumference of the large circle of the Observatory. For each point of the Observatory we thus construct the coordinate ψ. By subdividing the equatorial circle of the Observatory. we construct in the same way the angular coordinate φ.

We now send a pulse of light from the point $P(\varphi,\psi)$ on the Observatory to C. To all points reached by this pulse we then assign the coordinates φ and ψ of the point $P(\varphi,\psi)$.

Construction of the Radial Coordinate R

In Euclidean Geometry the experimental relation between the measured circumference N of a grand circle and the measured distance to the center is given by $r = N / 2\pi$. We however do not assume Euclidean Geometry is valid for the space outside the body. To measure r we must drill a hole in the body small enough not to significantly disturb the spherical symmetry. Because for a star this is not possible we will not use r as the radial coordinate. For a point outside the body we can measure the circumference N as we did in constructing the angular coordinates. We are free to assign a radial coordinate R such that $R = N/2\pi$ and in this way generalize the radial coordinate r of Euclidean Geometry.

We can measure the circumference N(0) of the surface of the body in the usual way by laying rigid rods of 1 meter along a grand circle. To a point P(0) on the surface we then assign he radial coordinate $R(0) = N(0)/ 2pi$. From P(0) we vertically stack up rods of 1 meter to reach subsequently the points P(1), P(2), P(3), ….. and so on until we reach the point P on the Observatory. We assume a rod not deforms under its own weight.

We can imagine that a "massless dread" connects the point P(0) to the point P. On that dread we mark at P(0) the number 0 and going up we mark the increasing numbers 1, 2, 3 , … and so on for the heights of the points P(1), P(2), P(3), and so on. These points define new spheres. Of these spheres we measure the circumferences and so determine R(1), R(2), R(3), and in this way so construct the radial coordinate R.

In constructing ψ we found that to a rod of 1 meter corresponds an angular distance $d\psi(1\text{meter}) = 2\pi/N = 1/R$.

We remark that a rod at rest on a supporting platform in a gravitational field can deform under its own weight. Although we can determine by experiment the degree of that deformation by applying mechanical forces to the rod in the laboratory , we will here ignore any such deformation. By marking on that "massless dread" the successive positions of the rod and going from one marking to the next, we avoid to assemble a multitude of vertical rods pressing each other down cumulatively.

Construction of the Time-coordinate t

In SR we construct the time coordinates by the principle of equal return-time by sending at an instant we call $t = 0$ a pulse of light from a point P to a point Q over a distance of 1 meter. To the instant of arrival in Q we then assign the $t = 1/c$. When the pulse instantly reflects we assign to the instant of return in P the time $t = 2/c$. By repeating this process and extending to other points at distances of 1 meter the time-coordinates for all points are assigned. To intermediate points linearly interpolated values are assigned.

For the massive spherical symmetric body we use the same principle. At the point P of the Observatory we let a pulse of light repeatedly be reflected over a vertical upward distance of 1 meter. These pulses return in P after a time-interval we call 2/c because at the Observatory SR is assumed to be valid.

Now we seek experimentally the point P_1 vertical below P such that a reflected pulse from P_1 returns in P at the same instant as the reflected pulse send upward to Q. We assume this can be done with infinite accuracy. Then we seek the point P_2 below P_1 such that a reflected pulse from P_2 returns in P_1 at the same instant as the reflected pulse from P. We continue this process downward. In this way we select the points P_1, P_2, P_3, ... and so on.

Then at an instant we call t = 0 we start the assignment procedure at P by sending reflecting pulses from P to Q and P_1. We assign to the successive arrivals in P1 the time-coordinates 1/c, 3/c, 5/c, ... and so on. In the same way we assign to the successive arrivals in P_2 the time-coordinates 2/c, 4/c, 6/c, ... and so on. We assign accordingly the time-coordinates to the points P_3, P_4 , P_5, ... and so on. So "it takes time" to construct the time-coordinates.

See *figure 1* for the time-assignments to the points P_n, P_{n+1} and P_{n+2}.

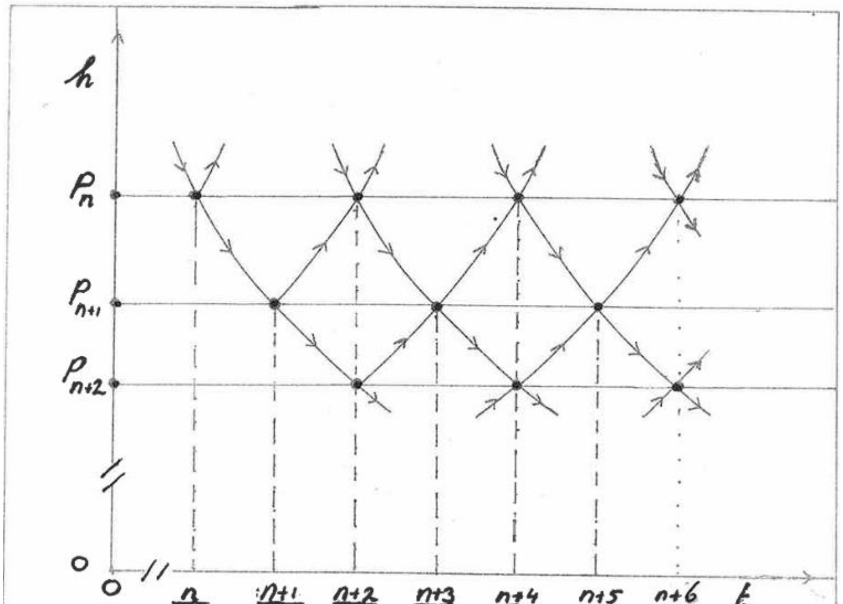


figure 1

In this way we reach discrete points. To refine the assignment we determine experimentally the intermediate points for which upward and downward pulses of light return at the same instant. To such points we then assign the time-coordinates (n + 0,5)/c, (n + 1,5)/c, (n + 2,5)/c ... and so on.

For this construction to be valid for points on all vertical lines we must synchronize the time t = 0 at the Observatory in the usual way by sending horizontal reflecting pulses of light along the surface of the Observatory.

The Metrical Functions

For two point-events (R, ψ, φ, t) and (R+dR, ψ+dψ, φ+dφ, t + dt) we have for $ds^2 = g_{44}\, dt^2 + g_{11}\, dR^2 + g_{22}\, d\psi^2 + g_{33}\, d\varphi i^2 + 2\, g_{14}\, dt\, dR + 2\, g_{24}\, dt\, d\psi + 2\, g_{34}\, dt\, d\varphi + 2\, g_{12}\, dR\, d\psi + 2\, g_{13}\, dR\, d\varphi + 2\, g_{23}\, d\psi\, dr$.

g_{14}, g_{24}, g_{34}, g_{12}, g_{13} and g_{23} are identical to zero

We will prove this for g_{14} by sending a reflecting pulse of light over dR from P to Q vertically above. So dψ = 0 and dφ = 0. By construction the time-intervals dt for both trajectories are equal.

For the trajectory PQ $ds^2\,(PQ) = g_{44}(R)\, dt^2 + g_{11}(R)\, dR^2 + 2\, g_{14}(R)\, dt\, dR$. For trajectory QP $ds^2\,(QP) = g_{44}(R+dR)\, dt^2 + g_{11}(R+dR)\, dR^2 - 2\, g_{14}(R+dR)\, dt\, dR$.

From SR we know that with respect to the cabin-coordinates $ds^2 = 0$ for both trajectories and so $ds^2(PQ) = ds^2(QP)$. For small dR we have in good approximation $g_{44}(R) = g_{44}(R+dR)$, $g_{11}(R) = g_{11}(R+dR)$ and $g_{14}(R) = g_{14}(R+dR)$. This leads to $g_{14} = 0$.

$$g_{22} = - 1/c^2\, R^2$$

To prove this, we position a rod of 1 meter with endpoints A and B tangential to a grand circle through the poles. For A and B then dR = 0 and dfi = 0. We chose dt = 0 by sending from the center of the rod opposite pulses to A and B. For the arrivals at A and B then $ds^2 = g_{22}\, d\psi(1 \text{ meter})^2 = g_{22} / R^2$ in which dψi(1meter) as before means the angular interval determined by A and B.

For a cabin oriented with its x-axis tangent to the grand circle we have for the rod of 1 meter $ds^2 = - 1/c^2\, dx^2 = - 1/c2$. By comparing we see $g_{22} = - 1/c^2\, R^2$.

$$g_{33} = - 1/c^2\, R^2 \sin^2\psi$$

We call a circle with ψ = constant a radial circle. At the sphere of the Observatory the circumference of a radial circle with coordinate ψ is smaller than the circumference of the equator by a factor sinψ. Due to the spherical symmetry that factor must remain valid for all spheres. For the sphere at R we have for the circumference of the equator N(equator) = 2π R. So the circumference of the radial circle at ψ equals N(ψ) = 2π R sinψ. Because also N(ψ) dφ(1meter) = 2π we see dφ(1 meter) = 2π/ N(ψ) = 1/(Rsinψ) in which dφ(1 meter) is the angular distance corresponding to the rod positioned tangent to the radial circle at the coordinate psi. So we have $ds^2 = g_{33}\, d\varphi(1\text{meter})^2 = g_{33}/ (R\sin\psi)^2$. With respect to the cabin $ds^2 = - 1/c^2$. By comparing $- 1/c^2 = g_{33}\, d\varphi(1 \text{ meter1})^2 = g_{33} / (R \sin\psi)^2$, so $g_{33} = - 1/c^2\, (R\sin\psi)^2$.

g_{44} and g_{11}

As a consequence of the spherical symmetry g_{44} and g_{11} are functions only of R. For a pulse of light tangent to a large circle through the poles (dR = 0 and dφ = 0) we have $ds^2 = 0 = g_{44}\, dt^2 - 1/c^2\, R^2\, d\psi^2$ and thus $g_{44} = 1/c^2\, R^2\, (d\psi/ dt)^2$ and so g_{44} is a positive function.

For a vertical pulse of light (dψ = 0 and dφ = 0) $ds^2 = 0 = g_{44}\, dt^2 + g_{11}\, dR^2$ and so $g_{11} = - g_{44}\, (dt/dR)^2$ and then g_{11} is a negative function. For mathematical convenience we introduce $g_{44} = e^v$ and $g_{11} = - 1/c^2\, e^w$ and so:

$$ds^2 = e^v\, dt^2 - 1/c^2\, e^w\, dR^2 - 1/c^2\, R^2\, d\psi^2 - 1/c^2\, R^2 \sin^2\psi\, d\varphi^2.$$

In Riemannian Geometry there exists the so called Ricci-tensor. This tensor consists of 16 functions R_{mn} of which 10 are independent because $R_{mn} = R_{nm}$. The functions R_{mn} are composed by the $g_{\mu\nu}$ and their first and second partial derivatives and so they are functions only of R. As a generalization of the conservation laws for energy and momentum of Classical Mechanics it is assumed in GR that in the empty space outside the body all these R_{mn} must be set equal to 0. Calculating these R_{mn} and equating them to 0 we find straightforward [6] four equations in which the accents mean differentiation with respect to R.

$$R_{44} = 0 \text{ gives } 2 v'' + 4 v'/R - v'w' + (v')2 = 0 \quad (1)$$

$$R_{11} = 0 \text{ gives } 2 v'' - 4 w'/R - v'w' + (v')2 = 0 \quad (2)$$

$$R_{22} = 0 \text{ gives } R (v'-w') - 2 e^{w} + 2 = 0 \quad (3)$$

$$R_{33} = 0 \text{ gives } R (v'-w') - 2 e^{w} + 2 = 0 \quad (4)$$

The other six R_{mn} turn out to be identical to 0. Equations (3) and (4) are identical. Solving the three equations gives $e^{v} = 1 - \alpha/R$ and $e^{w} = 1 / (1- \alpha/R)$. And so $g_{44} = 1 - \alpha/R$ and $g_{11} = - 1/c^2 \, 1/(1- \alpha/R)$. In these α is a constant of integration that is characteristic of the assembled body.

This solution is mathematically not valid for R = α. For R = α there turns out to be a coordinate-singularity where the construction of the coordinates as described apparently fails. So, for R > α the Schwarzschild-solution is

$$ds^2= (1- \alpha/R)\, dt^2 - 1/c^2 \,[1/(1- \alpha/R)]\, dR^2 - 1/c^2 R^2 d\psi^2 - 1/c^2 R^2 \sin^2\psi \, d\varphi^2$$

It is remarkable that the endless varieties of static spherical symmetric bodies have a similar formula for the metrical functions. They only differ in the numerical value of the parameter α. In Newton's Law of Gravity, the description of the gravitational field of these bodies also differ in the numerical value of the one parameter we know as the mass M of the body. This indicates that α and M are related.

Analytical Relation between dR and dh

We have already determined the experimental relation between R and h by measuring the circumference N = 2π R of the large circle for each height h. From the metric we find the analytical relation between R and h.

At point P with radial coordinate R we let the cabin be oriented with its z - axis in the vertical direction. From P we send vertical pulses of light up and down such that the reflected pulses return in P at the same instant. So, by the construction of the time-coordinate these pulses arrive in the points up and down simultaneous in t. For the space interval dR between these two points we then have $ds^2 = - 1/c^2 \,[1/(1-\alpha/R)]\, dR^2$.

With respect to the cabin we have $ds^2 = -1/c^2 \, dh^2$. By comparison we then have $[1/(1-\alpha/R)]dR^2 = dh^2$ and so $dR = \surd(1-\alpha/R)\, dh$ or $dh = 1/\surd(1-\alpha/R)\, dR$. The relation between R and h then is given by integrating dh from 0 to h.

Definition of the Second and the Numerical Value of c

The second as the historical unit of time is defined on the surface of the Earth as the 1/24 x 1/60 x 1/60 part of the day. Had mankind habituated an other planet there would have been defined a different unit. The second is counted by devices called clocks co-moving with the Earth. With respect to the fixed stars the Earth is rotating around her own axis in her orbit around the Sun and her movement is also influenced by the moon and the other planets. Thus, the second defined this way cannot be considered to be stable.

Let us assume a clock positioned at the North-Pole on a platform counter-rotating the rotation of the Earth so the clock is at rest with respect to the fixed stars. By choosing the North-Pole a reflecting vertical pulse of light returns at the North-Pole and so we eliminate complications due to the rotation of the Earth. The rotation of the Earth with respect to the platform then defines the duration of a day. We assume that the Earth is spherical symmetric in form and constitution and we assume we can ignore a possible influence of the rotation of the Earth on the metrical functions. Considering the Earth is free-falling in the solar-system we assume we can ignore the influence of the gravitational fields of the other masses in the solar-system. With these assumptions we treat the Earth as a static spherical symmetric body and apply the Schwarzschild-metric.

With the clock we measure the return-time in seconds for a reflecting vertical pulse of light over a distance of 1/2 meter to be dT = 1/(3.0 x 108) second. This means the clock registers (3.0 x 108) x (24 x 60 x 60) reflections in a day. The value dT = 1/(3.0 x 108) is measured with respect to the cabin at the North-Pole. This value dT over a distance of 1/2 meter is the same for all local cabins and thus also for a cabin at the Observatory. At the Observatory we called dT = 1/c. From this follows c = 3.0 x 108 meter per second. For the definition of the second not to depend on the various interactions in the solar system and still to be related to the historical definition, we define the numerical value of c to be 3.0 x 108 by convention.

Measuring the Constant α for the Earth

The value of α can be determined by measuring the acceleration of a test-body free-falling from rest. It is assumed in GR that the path of a free-falling test-body is described by a geodesic line with the appropriate boundary conditions. Following G. C. McVittie [6] we have for a test-body free-falling from rest $d^2R/dt^2 = - \alpha c^2/(2R^2)$ for values of α/R << 1. From this we then have $\alpha = - 2 R^2/c^2 \, d^2R/dt^2$. The practical measurement of d^2R/dt^2 for such a test-body is described in appendix 1 and leads to α= 0,89 cm.

But there is a more direct geometrical way to measure α by using the relation dR = √(1- α/R) dh. For the surface h = 0 and N(0)= 40,2 x 10^6 meter. For a large circle at h = 1 then N(1) - N(0) = 2π R(1) - 2π R(0) = 2π [R(1) - R(0)] = 2π √(1 - α/R(0). From this we then determine α = (1/2π) N(0) - (1/2π) N(0) [N(1) - N(0)]2. If we could measure N(0) and N(1) with sufficient precision we would find N(1) = N(0) + 2π - 4,4 x 10^{-9} corresponding to α= 0,89 cm. In Euclidean Geometry N(1) = N(0) + 2π. We see that the deviation from Euclidean geometry amounts to only 4,4 nm. It is clear that this deviation is way too small to be measured in practice.

Gravitational Time-Dilatation

Two Identical light-clocks positioned at h_1 and h_2 tick at different rates when these rates are compared at the same height by sending pulses from one light- clock to the other. The upper clock at h_2 turns out to tick faster in a gravitational field.

See figure 2.

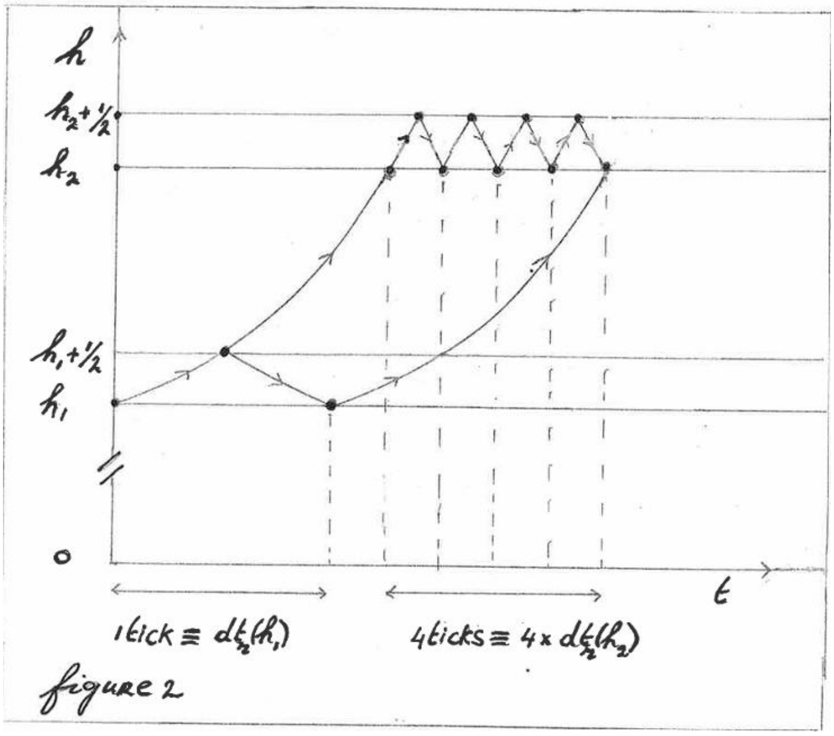


figure 2

We use light-clocks with a distance between the mirrors of 1/2 meter. We let the mirrors be semi-transparent so the pulses also can travel towards each other for comparison. The return-time $dt_r(h)$ for the reflecting pulse within a clock at height h is registered at the lower mirror of that clock. For two successive ticks $ds^2 = [1 - \alpha/R(h)]\, dt_r(h)^2$. With respect to the cabin coordinates $ds^2 = dT_r^2 = 1/c^2$ for these ticks. By comparison we then have that $dt_r(h) = (1/c) / \surd[1 - \alpha/R(h)]$. More in general for a point the relation between dt and dT of the local cabin is thus given by $dt = dT / \surd(1- \alpha/R)$.

For $\alpha << R(h)$ we have in good approximation $dt_r(h)= (1/c)\, [1 + \alpha/2R(h)]$. It then takes the number of $F(h) = 1/dt_r(h)$ ticks to define a time-interval of 1 second. Thus, we have $F(h) = 1/ \{[(1/c)\, [1 + \alpha/2R(h)]\} = c - \alpha c/2R(h)$. For the clock at h_1 we thus have $F(h_1) = c - \alpha c/2R(h_1)$ while for the clock at h_2 we have $F(h_2)= c - \alpha c/2R(h_2)$. We notice that for a light-clock at near- infinity we have c ticks in 1 second as in SR.

The successive pulses reflecting at the upper mirror of a clock at h_1 are partly transmitted and travel upward to an identical clock at h_2. The reading of the clock at h_1 can thus so to say be observed at h_2 by " looking down". At the arrival of the first transmitted pulse at h_2 we let the clock at h_2 start ticking. We let that clock stop at the arrival of the $F(h_1)$-th transmitted pulse from below. As a consequence of the construction of the time-coordinate the arrivals of the first and $F(h_1)$-th pulse at h_2 define at h_2 the same time-interval of 1 second.

For $h_2 - h_1 << R$ we have $R(h_2) = R(h_1) + \surd[1- \alpha R(h_1)]\, (h_2 - h_1)$. So, in good approximation we have :

$$F(h_2) = c - \alpha c/2R(h_2)$$

$$= c - (\alpha c/ 2\{R(h_1) + \surd\, [1- \alpha R(h_1)]\, (h_2 - h_1)\}$$

$$= c - \alpha c/2R(h_1) + [\alpha c/2R(h_1)^2]\, (h_2 - h_1).$$

Because $F(h_1) = c - \alpha c/2R(h_1)$ we have $F(h_2) = F(h_1) + [\alpha c/2R(h_1)^2]\, (h_2 - h_1)$.

We see that the light-clock at h_2 ticks faster than the light-clock at h_1 as observed at h_2 by $[\alpha c/2R(h_1)^2]$ (h_2-h_1) ticks per second.

We remark that the working of other types of clocks can not be calculated using GR. In case of two mechanical identical pendula the one at h_2 ticks obviously slower than the one at h_1 for $h_2 > h_1$ according to the classical law of the pendulum. Therefore, to determine the precise behavior of other clocks in a gravitational field we should compare their readings with the reading of the light-clock.

Gravitational Redshift

We treated the emission and reflection of a pulse of light as in geometrical optics of classical physics. According to modern physics light can be described as photons emitted from atoms. For simplicity we assume the emitting atom is at rest and emits only photons of one spectral line to exclude the contribution of the Doppler effect and avoid difficulties in identifying the spectral-line among other spectral lines.

We assume a photon consists of one wave with a front f and a rear r. With respect to the cabin of an emitting atom we call dT_e the time-interval between the emission of f and r.

With respect to that cabin f travels with velocity c so for the wavelength λ_e of the emitted photon we have $\lambda_e = c\ dT_e$. The wavelength λ_e is measured by interferometry. We further assume λ_e to be independent of the height of the emitting atom.

See figure 3.

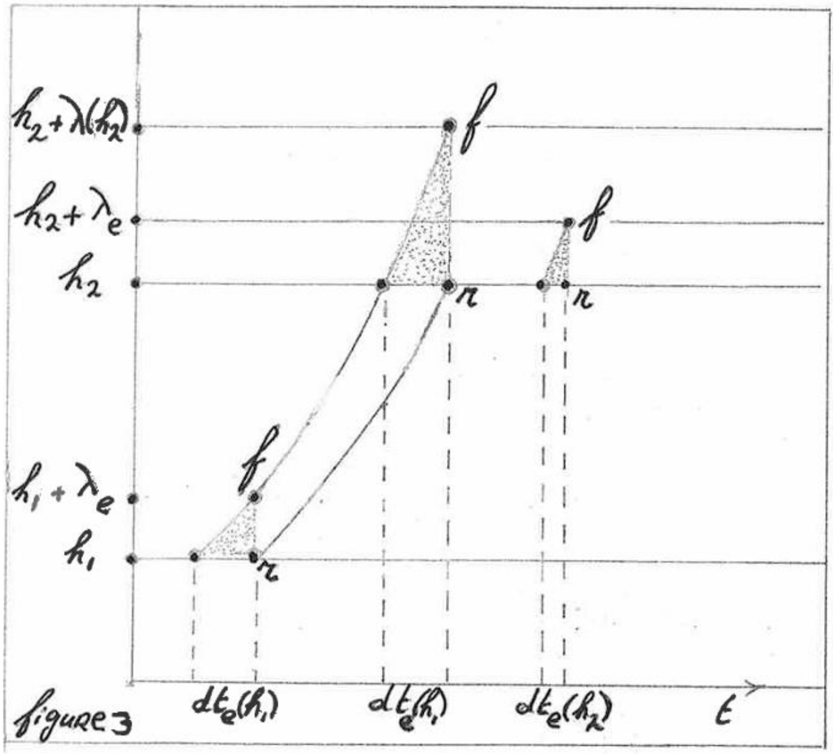


figure 3

From $dt = dT\ /\surd(1 - \alpha/R)$ we have for the time-interval dt_e with respect to the constructed time-coordinate $dt_e = dT_e\ /\surd(1-\alpha/R) = (\lambda_e/c)\ /\ \text{root}(1-\alpha/R)$. We see that dt_e depends on R and thus on h. So, we write $dt_e(h)$.

For the emission of a photon at h_1 we then have $dt_e(h_1) = (\lambda_e /c) /\surd[1-\alpha/R(h_1)]$ and so we have $\lambda_e= c\, dt_e(h_1) \surd[1-\alpha R(h_1)]$. For the emission of a photon at h_2 we find in the same way $\lambda_e = c\, dt_e(h_2) \surd[1-\alpha/R(h_2)]$.

By comparison then $c\, dt_e(h1) \surd[1-\alpha/R(h1)] = c\, dt_e(h2) \surd[1-\alpha/R(h2)]$.

At h_2 the time-interval between the arrivals of f and r of the photon emitted from the atom at h_1 remains $dt_e(h_1)$ by the construction of the time-coordinate. At the instant r arrives at h2 then f has already traveled upward according to the expression $ds2 = 0 = (1-\alpha/R(h_2)\, dt_e{}^2(h_1) - (1/c^2) \{1/ [1 - a/R(h_2)]\}dR^2$ and so we find that $dR = c\, dt_e(h_1)[1-\alpha/R(h_2)]$.

According to $dh = dR /\surd(1 - a/R)$ we then have $dh = c\, dt_e(h_1) \surd[1 -\alpha/R(h_2)]$. That distance dh between f and r must be equal to the wavelength of the upward moving photon arriving at h_2 and so $\lambda(h_2) = c\, dt_e(h_1) \surd[1 -\alpha/R(h_2)]$.

For a newly emitted photon from an identical atom at h_2 we have already found $\lambda_e = c\, dt_e(h_2) \surd[1-\alpha/R(h_2)]$.

For $\alpha<< R(h_1)$ and $h_2-h_1 << R(h_1)$ and using $dt_e(h_1) = (\lambda_e /c) /\surd[1-\alpha/R(h_1)]$ we then find in good approximation:

$$\lambda(h_2) - \lambda_e = c\, dt_e(h_1) \surd[1 -\alpha/R(h_2)] - c\, dt_e(h_1) \surd[1- \alpha/R(h_1)]$$

$$= [\alpha\, \lambda_e /2R(h_1)^2]\, (h_2 - h_1).$$

And so $\lambda(h_2) = \lambda_e + [\alpha\, \lambda_e/2R(h_1)^2]\, (h_2 - h_1)$. From this we see that $\lambda(h_2)> \lambda_e$.

We remark that the expressions for the gravitational time-dilatation and the gravitational redshift are closely related.

A STATIC AXIAL-SYMMETRIC BODY

Assembling the Body

As with the spherical body we can construct an axial-symmetric skeleton in Euclidean space and then fill that skeleton in an axial-symmetric way by sending in small masses from near-infinity. For simplicity we assume the constructed body has mirror-symmetry.

We will in this case construct the so-called “ canonical cylinder-coordinates” introduced by H. Weyl [8]. These coordinates are a generalization of the cylindrical coordinates from Euclidean Geometry. Therefore, we use a cylindrical Observatory at near-infinity.

Construction of the Angular Coordinate φ

We start the construction of the angular coordinate φ in the equatorial plane of the body. We divide in that plane the circle at the observatory in N equal parts with the rod of 1 meter, thus measuring its circumference N. Then a rod of 1 meter defines an angular interval of $2\pi/N$. Now we chose a point P at that circle and assign the value $\varphi = 0$ to that point P and assign the angular coordinate φ to the points of that circle in the usual way. By sending from P pulses of light in all directions rectangular to the circle, we assign the coordinate $\varphi = 0$ to all the points reached by these pulses. We can imagine the space-time outside the body to consist of a massless “canvas” on which we “paint” the angular coordinates φ for each point.

In reality there is no such canvas, but a point in space can be localized by the meeting of two pulses of light send from the Observatory. The concept of a "canvas" however can help to visualize the description.

Construction of the Radial Coordinate r

We are free to assign to a point P in the plane $\varphi = 0$ the radial coordinate r to be equal to the circumference N of the radial circle through P divided by 2π, so $r = N/2\pi$. This assignment differs from the assignment of the radial coordinate R in the spherical symmetric case because there we defined $R = N/2\pi$ for a large circle through P. Only for points in the equatorial plane the assignment corresponds. To a rod of 1 meter along the radial circle corresponds an angular distance $Nd\varphi(1\text{ meter}) = 2\pi$. From this $d\varphi(1\text{meter}) = 2\pi/N = 1/r$.

After assigning the radial coordinate to many points in the plane φ we connect the points with equal r . This gives a series of more or less "vertical" lines of equal r. In practice we must interpolate having determined the radial coordinate for only a finite set of points. So we must make the set of points large enough for the interpolation to be accurate. To the points on the axis of symmetry we assign the radial coordinate r = 0.

Construction of the Coordinate z

We construct in the plane φ the lines of equal z such that the lines of equal r and equal z are everywhere perpendicular to each other. This can consistently be done as stated by H. Weyl in his 1917 paper [8]. For the construction we can use a rectangular with rectangular arms of 1 meter. These lines are the more or less "horizontal" lines of constant z. We are still free to assign numerical values for z to these lines of equal z.

See figure 4.

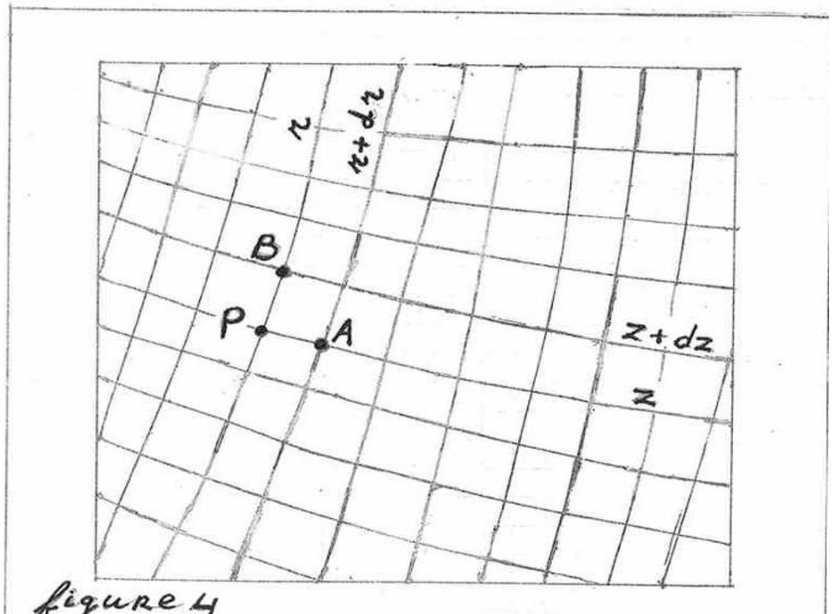


figure 4

We position the rectangular at a point P (r,z) tangent to the line of equal r. At the point A of the rectangular we read r + dr. We assign to the point that concurs with the point

B of the rectangular the coordinate z +dz with dz = dr. This value is then assigned to all points on the line of equal z through P. We start this procedure by assigning to the equatorial plane the value z = 0.

We remark that this construction corresponds in Euclidean Geometry to the construction of Cartesian coordinates. Then for all points dr = dz = 1.

Construction of the Time Coordinate t

As with the spherical body we send reflecting pulses of light at intervals of time of 1/c from the observatory towards the body and use the method of equal return-time. We can for example start at the point A of the Observatory on the axis of symmetry and assign the time-coordinate to the points of the axis by sending reflecting vertical pulses of light.

From the points on the axis we extend the assignment to the points in the plane φ = 0 and the other planes following the same principle of reflecting pulses. We are free to "jump from point to point" in this procedure as we please, but the orderly way is to follow the lines of equal r and equal z. This means that, starting at a point on the axis, we choose the next point to be on the line of equal z. In case close to the body we must work our way around the body to reach the surface of the body or to reach the equatorial plane we will in addition follow the lines of equal r.

Determination of the Metrical Functions

Having constructed the coordinates, we seek to determine the metrical functions $g_{\mu\nu}$ with respect to these coordinates. Again, by convention $g_{\mu\nu} = g_{\nu\mu}$. We use the index 4 for t and the indices 1, 2 and 3 for r, z and φ As in the spherical case it can be shown that g_{12}, g_{13}, g_{14}, g_{23}, g_{24} and g_{34} are zero.

As in the spherical case we position the cabin such that its + x-axis points in the direction of + φ. For the rod of 1 meter we have with respect to the cabin $ds^2 = - 1/c^2$. For the rod dφ (1meter) = 1/ r and so $ds^2 = g_{33} / r^2$. By comparison $g_{33} = - 1/c^2 r^2$. We now position the cabin with its other axes along the lines of equal r and equal z and compare the expressions for ds^2 for the rod of 1 meter along these axis as we did above. From this $g_{11}= g_{22}$. This gives $ds^2 = g_{44} dt^2 + g_{11} dr^2 + g_{11}dz^2 - 1/c^2 r^2 d\varphi^2$.

Calculation of Rij setting Rij = 0 gives 5 equations. One of these equations is $\partial/\partial r\, g_{44}= 0$ and so g_{44} is constant along the lines of equal z. Because at infinity we assume $g_{44} = 1$ it follows $g_{44} = 1$ everywhere. From another equation then follows $\partial/\partial z\, g_{11} = 0$ and so g_{11} is constant along the lines of equal r. Because at infinity we assume $g_{11} = -1/c^2$ it follows $g_{11} = - 1/c^2$ everywhere.

So now $ds^2 = dt^2 - 1/c^2 dr^2 - 1/c^2 dz^2 - 1/c^2 r^2 d\varphi^2$. We recognize this as the trivial solution for a massless body with respect to the cylindrical coordinates of Lorentzian space-time. This solution describes the situation of a massless body for which we did not fill the skeleton we started with. From this we see that finding a solution of the equations of GR not necessarily means that solution describes the gravitational field of the body under investigation. To find at least a non-trivial solution of the equations of GR we must construct a different radial coordinate.

The Radial Coordinate r = √(fl)

For convenience of notation we re-write $ds^2 = g_{44}\, dt^2 + g_{11}\, dr^2 + g_{11}\, dz^2 + g_{33}\, d\varphi^2$ as $ds^2 = f\, dt^2 - 1/c^2\, w\, dr^2 - 1/c^2\, w\, dz^2 - 1/c^2\, l\, dfi^2$ with f, w and l being positive functions of the coordinates r and z still to be constructed.

For a rod of 1 meter positioned at a point P tangential to its radial circle with respect to the cabin $ds^2 = -\, 1/c^2\, dx^2 = -\, 1/c^2$. Also, $ds^2 = -\, 1/c^2\, l\, d\varphi(1meter)^2$. From this $l = 1/d\varphi(1meter)^2$ and $d\varphi(1meter) = 1/\surd\, l$. So, we determine by measuring dφ(1meter) the value of l at the point P.

For a pulse of light emitted from P tangential to the radial circle we have that $ds^2 = 0 = f\, dt^2 - 1/c^2\, l\, d\varphi^2$ and so $f = 1/c^2\, l\, d\varphi^2/dt^2$. If we choose dt = 1/c second we find $f = l\, d\varphi(1/c\ second)^2$. So $d\varphi(1/c\ second) = \surd(f/l)$.

In this way we can for each point P determine f and l. Before having constructed the coordinates r and z we do not have f and l as mathematical functions. But f and l are physical entities that depend on the position in the gravitational field.

H. Weyl [10] choose r = √ (fl) as the radial coordinate and so $l = r^2/f$. This gives for the metric $ds^2 = f\, dt^2 - 1/c^2\, w\, dr^2 - 1/c^2\, w\, dz^2 - 1/c^2\, (r^2/f)\, d\varphi^2$. Now we have only two unknown functions f and w because l is replaced by r2/f. We may expect this simplifies the mathematical problem of solving the equations of GR.

H. Weyl made the choice r = root (fl) in an abstract mathematical context but did not give an explicit physical motivation for this choice. Regarding this choice, which looks rather arbitrary at first sight, we remark that r = root(fl) is in any case physically defined by the measured values of f and l.

We can write $d\varphi(1meter) = 1/\surd\, l = \surd\, f\, /\, \surd\, (fl) = \surd f\, /\, r$ and from this we see that r dφ(1 meter) = root f. In Lorentzian space-time we have f = 1 and then r dφ(1meter) = 1. In that case we see that r corresponds to the euclidean radial coordinate "radius". From this we see that the chosen definition of r is a generalization of the euclidean definition "radius".

But in any case, we are free to define r = √(fl) and investigate whether this leads to a solution of the equations of GR. With this definition we can construct the coordinates r and z in the same way as described above.

Determination of the Metrical Functions

We write for mathematical convenience $f = e^u$ and $w = e^v$. This gives for the expression $ds^2 = e^u\, dt^2 - 1/c^2\, e^v\, dr^2 - 1/c^2\, e^v\, dz^2 - 1/c^2\, (r^2/e^u)\, d\varphi^2$.

We now calculate the Rij and set Rij = 0. This results in the following equations :

$$\partial^2u/\partial r^2 + \partial^2u/\partial z^2 + 1/r\, \partial u/\partial r = 0\ \dots\ (1)\quad (\partial u/\partial r)^2 - (\partial u/\partial z)^2 - 2/r\, [\partial(u + v)/\partial r] = 0\ (2)$$

$$r(\partial u/\partial r)(\partial u/\partial z) - \partial(u+v)/\partial z = 0 \qquad (3)$$

These equations do not give one explicit solution as in the static spherical symmetric case. Equation (1) we recognize mathematically as the Newtonian gravitational potential equation. From Newtonian physics we know there are as much mathematical functions u that fulfill this potential equation as there are massive bodies. For each of these functions u we can solve ∂v/∂r from equation (2). From (3) we then find ∂v/∂z. From ∂u/∂r, ∂u/∂z,

$\partial v/\partial r$ and $\partial v/\partial z$ we can determine by integration the functions u and v. In this way we can find a limitless number of solutions of the equations of GR in terms of the canonical coordinates r and z. See also [10].

Given the limitless number of solutions it remains to select from them the one that describes the gravitational field of the constructed body. To that end we must for each point (t,r,z,φ) verify that the metrical functions correspond to the experimentally determined values within the experimental accuracy.

Relation between the Schwarzschild-coordinates and the Canonical-coordinates

A static spherical-symmetric body is also axial-symmetric. By this we can derive the relation between the canonical-coordinates (t, r, z, φ) and the Schwarzschild-coordinates (t, R, ψ, φ). The derivation is given in appendix 2.

With these relations we can find the metrical functions for a static spherical body with respect to the canonical coordinates by transformation of the metrical functions of the Schwarzschild-solution by the usual tensor-transformation. These metrical functions become rather complicated.

A STATIONARY ROTATING AXIAL-SYMMETRIC BODY

Assembling the Body

We construct the body by letting a massless axial-symmetric skeleton rotate in euclidean space and fill the compartments of the skeleton in an axial-symmetric manner by bringing in small masses from a large distance. We can also set a solid axial-symmetric body in rotation by mechanical means. In doing so there will be generated tensions in the body which will settle in time. We treat for simplicity only a stationary body with mirror symmetry. Mirror symmetry generally speaking conforms to astronomical reality.

We must keep in mind that the concepts form and rotation are used in a more or less intuitive way based on notions carried over from euclidean geometry. To describe the form and the rotation of the body we need to know the metrical functions with respect to the coordinates to be constructed.

Construction of the Angular Coordinate φ

For intuitive reasons we choose the angular coordinate φ to be "at rest with respect to the fixed stars" or the Observatory. Because the body rotates and different layers of the body can rotate at different rates, we first of all must define what we mean by the concept "at rest with respect to the fixed stars" or the Observatory.

We can send from a point on the cylindrical observatory an inward pulse inward and let that pulse meet another pulse send from another point on the observatory. The meeting point Q can so to speak be "painted on the canvas" of the space-time outside the body. If we repeatedly send identical pulses they will meet again and again at Q. That point Q we define to be at rest.

From the point P on the observatory located on the axis of symmetry we send repeatedly in an axial-symmetric way N pulses to the equatorial plane of the rotating body. We let the direction of these pulses be at some fixed angle with the axis thus forming a cone at departure. These pulses arrive at the equatorial plane at a circle. The angular distances between the points of arrival we define as $2\pi/N$. The same applies to all radial circles reached by these pulses. Varying the angle of the cones of emission will define different radial circles. We can “paint” these circles on the "canvas" of the space-time outside the rotating body. The points of these circles then are fixed geometrical entities at rest with respect to the fixed stars.

We now position a cabin to be at rest at some point E in the equatorial plane. We certainly must apply forces to let the cabin stay at rest at E to prevent the cabin from falling towards the body. We must also prevent the cabin from possibly being pulled in some side-way direction. We can imagine this cabin to be concurring momentarily with an identical cabin free-falling in the gravitational field and reaching its apex at rest in E.

Inside this cabin we have a near-infinitesimal part of the radial circle through E and its neighboring radial circles . By positioning in E the rectangular with arms of 1 meter tangential to the circle we connect E with the corresponding points on neighboring radial circles. At these points we repeat this procedure. In this way we can jump from radial circle to radial circle in a well-defined manner. To the points thus reached we assign the value $\varphi = 0$. For the N-1 other points on the radial circle through E we follow the same procedure and assign to these the values $\varphi = 2\pi/N$, $\varphi = 2x\ 2\pi/N$, fi = $3 \times 2\pi/N$ and so on.

If we assume the body is like a rotating sphere or ellipsoid. Near the surface of the body then the cabin is at rest with respect to the fixed stars while the body is moving underneath the cabin. This corresponds to the intuitive state of affairs carried over from the euclidean description.

We note that this way of constructing the angular coordinate is only possible when we can by physical means keep the cabin at rest with respect the fixed stars because only then we can position the rod as described.

Constructing the Time-coordinate t

At the point P on the axis of the Observatory we send a repeatedly reflecting pulse of light upward over a distance of 1/2 meter. These pulses determine intervals of 1/c second.

Along the axis we now assign the time-coordinates to the points on the axis by sending reflecting pulses up and down the axis by the principle of equal return-time. By the axial-symmetry these pulses will not deviate from the axis.

We expand the assignment to the other points by sending reflecting pulses of light from the points on the axis into the plane $\varphi = 0$ iterating from point to point. In this we must take care the lines we follow do not intersect each other and remain in the plane $\varphi = 0$. Further we have complete freedom to go from point to point using the principal of equal return time.

Of pulses send from the axis into the plane $\varphi = 0$ we do not know a priori that they remain inside that plane. The pulse might deviate from that plane due to the rotation of the body. In Newtonian gravity this is assumed not to happen. This deviation of such a pulse actually occurs as is shown in appendix 4. For this reason, we must "force" the time-assigning pulses to remain within the plane φ= 0. By this we mean that the time-assigning pulses must reflect between two neighboring points of the plane φ = 0 and that we "re-direct" the assigning pulses from point to point. The path followed by a pulse between two finitely separated points in the plane φ = 0 may leave the plane in the space between, but for small distances we ignore that.

The Behavior of Rods and Pulses of Light Tangent to Radial Circles

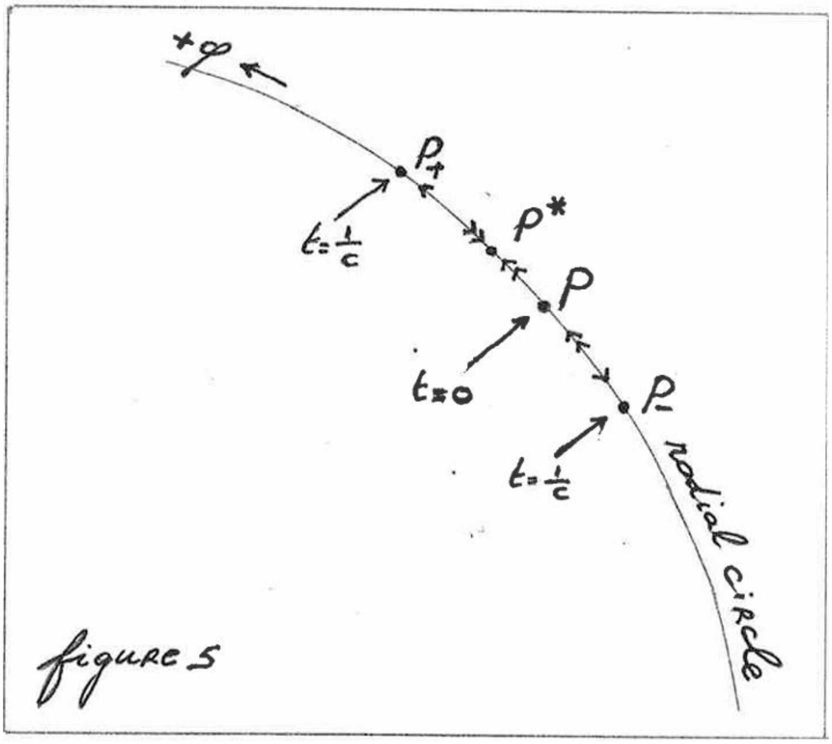
We now explore the behavior of rods and pulses of light along radial circles and will use that behavior to construct the coordinates r and z. For radial circles we have dr = 0 and dz = 0 and so it is not relevant we have not yet constructed r and z.

For a radial circle $ds^2 = g_{44}\, dt^2 + g_{33}\, dfi^2 + (g_{43} + g_{34})\, d\varphi\, dt$. If we introduce the function $q = (g_{43} + g_{43})\ /2$ then $ds^2 = g_{44}\, dt^2 + g_{33}\, df\varphi^2 + 2\, q\, d\varphi\, dt$. The functions g_{44} , g_{33} and q exist as physical entities dependent on position. They do not yet exist as mathematical functions of r and z because r and z are still to be constructed.

For notational convenience we write $ds^2 = f\, dt^2\, 1/c^2\, l\, d\varphi^2 + 2\, q\, d\varphi\, dt$. In case q = 0 we regain ds^2 as in the static axial-symmetric case, so q can be seen to relate to the rotational state of the body. As before f and l will have positive values. The sign of q will follow from experiment.

Sending Pulses of Light along a Radial Circle During dt= 1/c Second

See figure 5.



At the instant a time-assigning pulse arrives at a point P we send from P in the directions + φ and - φ a pulse of light tangent to the radial circle. These pulses meet the next time-assigning pulses arriving at the radial circle. The two meeting points we call P_+ and P. These points are by construction arrived at after dt = 1/c second and so these arrivals are simultaneous in t.

For the arrivals at P_+ and P_- $ds^2 = f\,(1/c)^2 - 1/c^2\, l\, d\varphi^2 + 2\, q\, d\varphi\,(1/c) = 0$ by comparison with the coordinates of the local cabin at rest in P. This is a quadratic equation in dφ. This equation has the solutions:

$$d\varphi_+ = qc/l + [\surd(lf + q2c2)] / l \quad d\varphi_- = qc/l - [\surd(lf + q2c2)] / l$$

In case q > 0 at P the angle $d\varphi_+ > 0$ and $d\varphi_- < 0$ with $d\varphi_+$ > modulo $d\varphi_-$. In case q < 0 we also have $d\varphi_+ > 0$ and $d\varphi_- < 0$, but now $d\varphi_+$ < modulo $d\varphi_-$. The sign of q at P will depend on the rotational state of the body. We can imagine by analogy with fluid-mechanics that in case all parts of the body rotate in the direction of +φ the "angular velocity" dφ/dt of the pulse in the +φ direction will (modulo) be larger than in the - φ direction. In the following we assume for simplicity q > 0.

Paying attention to the signs of $d\varphi_+$ and $d\varphi_-$ the angular distance between P_+ and P_- equals $d\varphi(P_+P_-) = (d\varphi_+) - (d\varphi_-) = 2\,[\surd\,(lf + q2c2)]/\,l$. When we let the pulses be reflected from P_+ and P_- they will meet at some point P*(dt, φ*).

From the axial-symmetry follows that the time-interval dt = 1/c to transverse PP^+ is equal to the time-interval to transverse P_-P^*. Also, it follows that the time-interval dt = 1/c to transverse P_+P^* must be equal to the time-interval to traverse PP_-. Then the time-interval for PP_+P^* and PP_-P^* equals dt = 2/c. We also have $\varphi^* = (d\varphi_+) + (d\varphi_-) = 2qc/l$. Because these relations also are valid for all values of dt < 1/c we can state that P* so to speak travels with an angular velocity $d\varphi^*/dt = [2qc/l\,] / (2/c) = qc^2/l$ along the radial circle.

We can imagine there is at P at the instant of emission a "free-falling cabin" moving tangential to the radial circle, so to speak a moving cabin within the cabin at rest. With respect to that moving cabin the two pulses are emitted and return in P* at the same instant. Therefore, these pulses must be assumed to be reflected simultaneous with respect to the coordinates of the moving cabin. T. Lewis already referred to the movement of such a cabin [11].

In appendix 3 we calculate the velocity of this moving cabin with respect to the cabin at rest to be v = (modulo) $qc^2/\surd(lf + q^2c^2)$.

Sending Pulses of Light Along a Radial Circle Over dφ

For pulses of light send from P over an angular distance dφ to F^+ in the +φ direction and F^- in the -φ direction we have for the points of arrival the expression $ds2 = f\, dt^2 - 1/c^2\, l\, d\varphi^2 + 2\, q\, d\varphi\, dt = 0$. Solving this quadratic equation in dt gives : $dt_1 = [- q + (1/c)\,\surd(lf + q^2c^2)]\, d\varphi/\, f$

$dt_2 = [- q - (1/c)\,\surd(lf + q^2c^2)]\, d\varphi\, /\, f$

For q > 0 and $dt^1 > 0$ we must have dφ> 0 and so we associate dt_1 with the pulse in the +φ direction. In the same way we associate dt_2 with the pulse in the -fi direction. So we write $dt_1 = dt_+$ and $dt_2 = dt_-$. We see $dt_+ < dt_-$ and so the points F^+ and F^- are not reached simultaneous in t.

After reflection the pulses return in P at the same instant $(dt_+ + dt_-)$ because of the axial-symmetry.

We remark that with respect to the coordinates of the cabin at rest in P the points F^+ and F^- are reached simultaneous because the pulses travel over the same distance and return at the same instant.

Measuring l, f and q

For the reflecting pulses of light send over a distance of 1 meter the return-time equals $dt(return) = (dt^+) + (dt^-) = 2/c\,[\surd(lf + q^2c^2)]\, d\varphi(1meter) / f$. For the emission and reception we then have $ds^2 = f\, dt(return)^2$ and so:

$$ds^2 = 4/c^2\,(lf + q^2c^2)\, d\varphi(1meter)^2/f.$$

With respect to cabin at rest in P we have ds2 = $4/c^2$. By comparison we then have $(lf + q^2c^2)\, d\varphi(1meter)^2 = f$.

At P we can measure $dt^+ = A$, $dt^- = B$ and $d\varphi(1meter) = D$ with $D > 0$ and so :

$$dt^+ = [-q + 1/c\,\surd(lf + q^2c^2)]\, D / f = A \qquad (1)$$

$$dt^- = [q + (1/c)\,\surd(lf + q^2c^2)]\, D / f = B \qquad (2)$$

$$(lf + q^2c^2)\, D^2 = f \qquad (3)$$

From these three equations we find straightforward:

$$l = 4\,AB / [D\,(A+B)]^2 \quad f = (4/c^2) / (A+B)^2$$

$$q = (2/c^2)\,(B - A) / [\,D\,(A+B)^2]$$

Constructing the Radial-coordinate r

In the static axial-symmetric case we constructed $r = \surd(fl)$. In 1931 T. Lewis [12] made for the radial coordinate the choice $r = \surd(fl + q^2)$. In this he used the value $c = 1$ for the velocity of light. He based this choice on a mathematical formalism as H. Weyl did in the static case. We note that for each point the values for l, f and q can be measured and so the value of r is well-defined.

For pulses of light send in opposite directions tangent to a radial circle over a time-interval $dt = 1/c$ from P to P^+ and P^- we know $d\varphi(P^+P^-) = (d\varphi^+) - (d\varphi^-) = (2/l)\surd(lf + q^2c^2)$. For the static case we have $q = 0$ and so $d\varphi(P^+P^-) = (2/l)\surd(lf)$. So in the stationary case $\surd(lf + q^2c^2)$ takes the place of $\surd(lf) = r$. Therefore, we can say that the definition $r = \surd(lf + q^2c^2)$, corresponding to the choice T. Lewis made, forms a generalization of the definition $r = \surd(lf)$ for the static case.

We now can determine by the measurement of l, f and q the value of r for the points in the plane $\varphi = 0$ and then construct the coordinate-lines for equal r by connecting the points with equal value of r as we did in the static case. To the axis of symmetry, we assign $r = 0$.

By the choice of r the experimental values of f, l, w and q of course do not change. Only the mathematical functions corresponding to f, l, w and q will be transformed as remarked earlier.

From $r = \sqrt{(lf + q^2c^2)}$ we have $l = (r^2 - q^2c^2) / f$ and so we can eliminate l from the expression of ds^2. For two neighboring point-events on a radial circle we then have the expression ds^2

$$= f\, dt^2 - 1/c^2\, [(r^2 - q^2c^2) / f]\, d\varphi^2 + 2\, q\, d\varphi\, dt.$$

Construction the Coordinate z

In case of mirror symmetry we assign to the points in the equatorial plane the value z = 0. We draw the lines of equal z as the lines that are with respect to the cabins at rest perpendicular to the lines of equal r as described in the static case. We assign the numerical values for the lines of equal z by determining dz using a T-square with arms of 1 meter as described in the static case.

Solving the Equations of GR

It can be shown that $g_{12} = g_{21} = g_{13} = g_{31} = g_{14} = g_{41} = g_{24} = g_{42} = 0$ and also $g_{11} = g_{22}$. For $g_{11} = g_{22}$ we write $- 1/c^2\, w$ with w positive everywhere. The expression for ds^2 then becomes $ds^2 = f\, dt^2 - 1/c^2\, w\, dr^2 - 1/c^2\, w\, dz^2 - 1/c^2\, [(r^2-q^2c^2)/f]\, d\varphi^2 + 2\, q\, d\varphi\, dt$ with f, w and q for as yet unknown mathematical functions.

We now can calculate the Rij and set Rij = 0. Because f and w are positive we can set for mathematical convenience $f = e^U$ and $w = e^V/e^U$. For q we write $q = (1/c)\, B\, e^U$ in which the sign of q depends on the sign of B. This leads to the following equations:

$$\partial^2U/\partial r^2 + \partial^2U/\partial z^2 + 1/r\, \partial U/\partial r = - 1/2\, [e^{4U}/r^2]\, [(\partial B/\partial r)^2 + (\partial B/\partial z)^2] \quad (1)$$

$$\partial[(\partial B/\partial r)\, e^{4U}/r]\, /\partial r + \partial[(\partial B/\partial z)\, e^{4U}/r]\, /\partial z = 0 \quad (2)$$

$$1/r\, \partial V/\partial r = (\partial U/\partial r)^2 - (\partial U/\partial z)^2 - [e^{4U}/4r^2][(\partial B/\partial r)^2 - (\partial B/\partial z)^2] \quad (3)$$

$$1/r\, \partial V/\partial z = 2\, \partial U/\partial r\, \partial U/\partial z - [e^{4U}/2r^2]\, \partial B/\partial r\, \partial B/\partial z \quad (4)$$

For B= 0 and thus q = 0 we regain the equations for the static body.

If we find functions U(r,z) and B(r,z) that obey (1) and (2) then V(r,z) can be calculated using equations (3) and (4) by integration [13]. It is obvious that the chance to find the mathematical functions that describe the gravitational field of the constructed body is not large because there is an infinity of possibilities for different layers of the body to be materially composed and to rotate. Given some solution we must for each point (t,r,z,φ) verify that the metrical functions correspond to the experimentally determined values within the experimental accuracy.

The Kerr-metric

H. Stephani [14] mentions the Kerr-solution as the best known and most important representative of the class of stationary axial-symmetric solutions. The Kerr-solution in canonical coordinates is given in [15]. For simplicity we limit the treatment to the equatorial plane, thus z = 0 and dz = 0. For that plane the solution is given by: $ds2 = f\, dt^2 - 1/c^2\, w\, dr^2 - 1/c^2\, [(r^2-q^2c^2)/f]\, d\varphi^2 + 2\, q\, d\varphi\, dt$ in which f, w and q are equal to : $f = (r^2 - S^2)/ [M + \sqrt{(r^2 + M^2 - S^2)}]^2$

$$w = [M + \surd(r^2 + M^2 - S^2)]^2 / (r^2 + M^2 - S^2) \quad q = - (1/c)\, 2\, S\, M / [M + \surd(r^2 + M^2 - S^2)]$$

M and S are parametric constants akin to the parametric constant α in case of the Schwarzschild-solution. In the Schwarzschild-solution the constant α is related to the Newtonian mass of the body. This suggests that one of the two parameters M and S might be related to the Newtonian mass of the body and the other to the Newtonian angular momentum of the body. In case both S = 0 and M = 0 we have f = 1, w = 1 and q = 0. So this case represents Lorentzian space-time for a massless body. In case S = 0 and M # 0 we have q = 0 and so we have a static axial-symmetric body with:

$$f = r^2 / [M + \surd(r^2 + M^2)]^2$$

$$w = [M + \surd(r^2 + M^2)]^2 / (r^2 + M^2)$$

This solution should encompass the Schwarzschild-solution in canonical coordinates. The relation between the radial-coordinate r and the Schwarzschild radial-coordinate R is given by $r = \surd(R^2 - \alpha R) \sin\psi$ and so in the equatorial plane $r = \surd(R^2 - \alpha R)$.

From this we have $f = (R^2 - \alpha R) / [M + \surd(R^2 - \alpha R + M^2)]^2$. In the Schwarzschild-solution we have $f = 1 - \alpha/R$. And so, for the Schwarzschild-solution the equation $(R^2 - \alpha R) / [M + \surd(R^2 - \alpha R + M^2)]^2 = 1 - \alpha/R$ must be fulfilled. By choosing M = a/2 this equation is fulfilled and $f = 1- \alpha/R$.

Because for a static spherical symmetric body α is related to the Newtonian mass of the body, we infer that M, also in case S # 0 , is related to the Newtonian mass of the body. From the fact that S = 0 corresponds to a static body we infer that S # 0 is related to the Newtonian angular momentum of the body.

COORDINATE - SINGULARITIES

In case of a static spherical symmetric body we determined the radial coordinate R by the measured circumference of a grand circle divided by 2π. The Schwarzschild-solution then is valid for all values of R > α. So for all these values the coordinates can be constructed. We now look for a physical explanation for the special situation at R = α.

For R = α the function $g_{44} = (1- \alpha/R) = 0$ and $g_{11} = - (1/c^2)\, 1/ (1- \alpha/R)= 1/ 0$ and so is undetermined. A pulse send over a distance of 1 meter returns with respect to a cabin at rest at R = α after dT = 2/c. That time interval thus corresponds to $ds^2 = 4/c^2$. That same time-interval should correspond to the Schwarzschild-coordinates to $ds^2 = 0\, dt^2$. These results are inconsistent. In constructing the time coordinate for a point, we assumed that it is possible for a vertically upward moving cabin to reach the apex of its path being at rest at that point. From the above result we infer that for R = α this is physically not possible.

In case of a stationary body there is also a different kind of singularity. We assumed that an upward moving cabin could reach its apex in such a way that there could exist a "cabin within the cabin" at rest. These two cabins move with respect to each other with a velocity $v = qc^2/ r$. In case r = qc that velocity reaches the value v = c which according to SR is not possible. From this we infer that for r = qc the construction of the coordinates fails. At least in the equatorial plane we can only construct the canonical coordinates for values of r > (modulo) qc. We infer that for smaller values of r a cabin cannot be maintained at rest. This singularity is called the "stationary limit".

This also follows in another way. We have $f = (r^2 - S^2)/ [M + \surd(r^2 + M^2 - S^2)]^2$ in the equatorial plane. From this a singularity occurs for r = (modulo) S because then f = 0. Then $q = - (1/c)\ 2 S M/ [M + \surd (r^2 + M^2 - S^2)] = - (1/c) S$. So, we have S = - qc. From this we see that for r = (modulo) qc a singularity occurs in accordance with the above result.

To describe the gravitational field in the area where our coordinates cannot be constructed, we need to choose other coordinates.

ON THE MATHEMATICAL FORM OF THE BODY

We define the mathematical form of the surface of a body by projecting the numerical values of the coordinates of the surface on R3. In SR a body with a spherical mathematical form at rest has with respect to a moving system of Lorentzian coordinates the mathematical form of an ellipsoid of revolution.

If we assume we meet no coordinate singularities on the way down to the surface of the body we can determine the coordinates of the points of the surface. In case of a spherical symmetric body we can construct both the Schwarzschild-coordinates and the canonical coordinates. With respect to the Schwarzschild-coordinates all points on the surface have the radial coordinate R(0) and so the projected surface in R3 is a spherical surface with "radius" R(0).

The relation between the Schwarzschild- coordinates (R, ψ) and the canonical-coordinates (r,z) is given by:

$$r (R, \psi) = R \surd(1 - \alpha/R) \sin \psi \quad z (R, psi) = (R - \alpha/2) \cos \psi$$

In the equatorial plane $\psi = \pi/2$ and so $r = R\surd(1 - \alpha/R) = R - \alpha/2+ ...$for small values of a/R. For the north-pole ψ= 0 and so r = 0 and z = R - α/2. Projecting these values in R3 we see that the form of the body is no longer a sphere.

In the limit for R going to α we see r = 0 at the equator and z = α/2 at the north-pole. We then have for the projected form of the surface a vertical rod of length α, corresponding to the findings of H. Weyl [16].

From this example we see that in GR , as in SR, the mathematical form of the body depends on the coordinates used.

In astronomy a photographic picture of the surface of a celestial body gives an impression of the form of the surface of that body. To know the mathematical form of the surface of the body we however need to construct the coordinates of the points at its surface. So, we must trace back the pulses of light composing the photographic picture. To trace back the paths of these pulses of light requires the metrical functions to be known.

SUMMARY AND CONCLUSION

The assemblage of static spherical- symmetric, static axial- symmetric and stationary axial-symmetric bodies is described. Then the a priori construction of coordinates by operational means in the space-time outside is described. For the construction of the coordinates we use reflecting pulses of light and small but finite rigid rods. In this way we determine experimentally the values of the metrical functions for each point. The construction leads to the expressions for ds2 with respect to the constructed coordinates. From these the mathematical expressions for the metrical functions can be found by solving the equations of GR. The a priori construction has the advantage that the symbols representing the coordinates have a physical meaning. In case we let these symbols for as yet physically undefined, we must somehow unveil their meaning a posteriori.

The a priori method makes the phenomena of gravitational time dilatation, gravitational redshift and the occurrences of coordinate singularities comparatively easy to understand because we know our position in the gravitational field and so to say have the constructed coordinates as a kind of "solid ground" to stand on. This gives the a priori construction didactical value.

We infer by the a priori method for a stationary axial-symmetric body that the canonical coordinates can only be constructed outside the "limit of stationarity". To learn more about the behavior of test-bodies, rods and pulses of light on the inner-side of a coordinate singularity other coordinates must be used.

The coordinates as constructed a priori describe the gravitational field before we have a mathematical solution of the equations of GR. Only in case the mathematical metrical functions agree within the limits of experimental accuracy to the experimental values do these functions describe the gravitational field of the constructed body.

REFERENCES

[1]. G. C. McVittie ; General Relativity and Cosmology , second edition (1965), 4.1

[2]. H. Weyl, "Zur Gravitationstheorie", Ann. der Physic 54 (1917), 117-145, page 117

[3]. Einstein, The Meaning of Relativity; Sixth edition (1956), page 86

[4]. A. Einstein , Jahrb. Rad. Elektr. 4, 411 (1907)

[5]. Pais, ' Subtle is the Lord ... ' (1982), chapter 9, pages 179 - 181

[6]. G. C. McVittie ; General Relativity and Cosmology ; 5.1

[7]. H. Weyl, "Zur Gravitationstheorie", Ann. der Physic 54 (1917), 117-145, page 117

[8]. H. Weyl, "Zur Gravitationstheorie", Ann. der Physic 54 (1917), 117-145, page 135

[9]. H. Weyl, "Zur Gravitationstheorie", Ann. der Physic 54 (1917), 117-145, page 135

[10]. H. Stephani, General Relativity, chapter 21.2.1, 1982

[11]. T. Lewis , 1931 , "Some special solutions of the equations of axial symmetric gravitational fields", chapter 3, page 181

[12]. T. Lewis , 1931 , "Some special solutions of the equations of axial symmetric gravitational fields", chapter 3, page 178

[13]. H. Stephani, General Relativity, chapter 21.2.2, 1982

[14]. H. Stephani, General Relativity (1982), chapter 24.1

[15]. G. C. Jones, J. E. Wang, Weyl card diagrams , Phys. Rev. D 71, 124019 (2005)

[16]. H. Weyl, "Zur Gravitationstheorie", Ann. der Physic 54 (1917), 117-145, page 140

[17]. G. C. McVittie ; General Relativity and Cosmology , second edition (1965), 2.3

APPENDIX 1: MEASUREMENT OF THE CONSTANT A FOR THE EARTH

The path of a test-body gives in good approximation $\alpha = -(2R^2/c^2)\, d^2R/dt^2$. To determine d^2R/dt^2 we let the test- body fall from h = 1/2 meter above the north-pole assuming an idealized Earth as described before.

See figure 6.

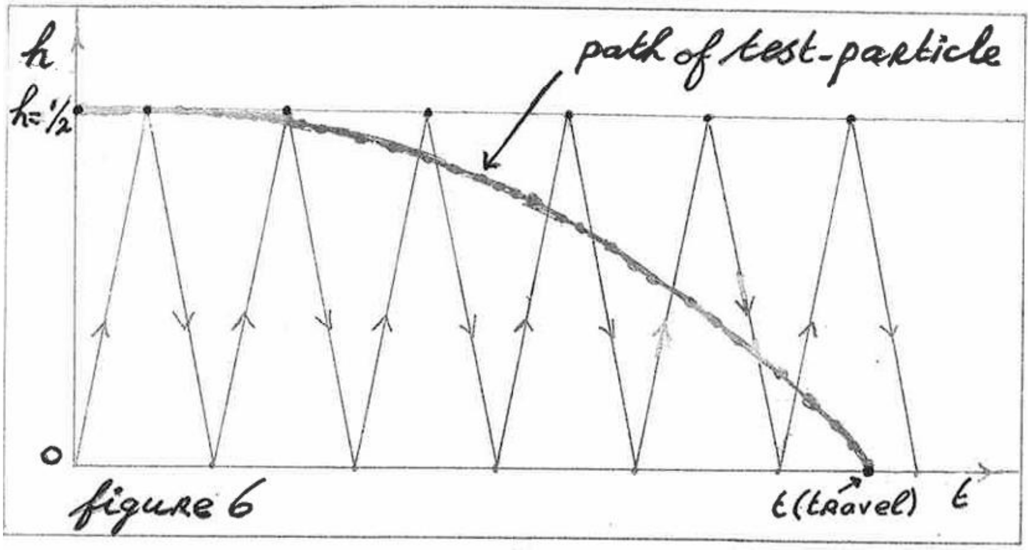


figure 6

At T = 0 we send a vertical pulse from the surface to the test-body and at the instant the pulse arrives at h we let the test-body fall. We let the pulse reflect repeatedly between the surface and h= 1/2 meter. At the instant the test-body hits the surface we stop the counting and read the number of reflections N. The systematic error in N is of order 1 due to the starting pulse and the circumstance that the final reflecting pulse generally not arrives at the surface at the same instant as the test-body.

For two successive arrivals at the surface $ds^2 = [1 - \alpha/R(0)]\, dt(\text{return})^2$. With respect to the cabin $ds^2 = dT(\text{return})^2 = 1/c^2$. By comparing we then find that $dt\,(\text{return}) = (1/c)/\sqrt{[1 - \alpha/R(0)]}$.

For each successive interval we measure dT(return) with respect to a new cabin and so for each interval dT(return) = 1/c. Then the measured time of travel T(travel) for the test-body to reach the surface equals T(travel) = N/c and so $t(\text{travel}) = (N/c)\,[1/\sqrt{[1 - \alpha/R(0)]}$.

From experiments near the surface of the Earth dh = (1/2) g T(travel)2 and so $T(\text{travel}) = \sqrt{(2\, dh/g)}$ in which g = - 9,8 m/s2. For dh = - 1/2 meter we then find $T(\text{travel}) = 1/\sqrt{(-g)}$. From T(travel) = N/c we then $N/c = 1/\sqrt{(-g)}$.

For the position R_t of the test-body at the time t we have by a Taylor-expansion : $R_t = R(1/2) + 1/2\,(d^2R/dt^2)\, dt^2 + \ldots$. At t = t (travel) we have $R_t = R(0)$ and so: $R(0) = R(1/2) + 1/2\,(d^2R/dt^2)\,(N/c)2\,\{1/\sqrt{[1 - \alpha/R(0)]}\}^2 + \ldots$

$R(1/2) - R(0) = -1/2\,(d^2R/dt^2)\,[1/(-g)] / [1 - \alpha/R(0)] + \ldots$

For dh = 1/2 meter also $R(1/2) - R(0) = (1/2)\sqrt{[1 - \alpha/R(0)]}$ and so : $(1/2)\sqrt{[1 - \alpha/R(0)]} = -1/2\,(d^2R/dt^2)\,[1/(-g)] /[1 - \alpha/R(0)]$ $d^2R/dt^2 = g / [1 - \alpha/R(0)]^{3/2} = g$ in good approximation.

From $\alpha = -2\,(R(0)^2/c^2)\,d^2R/dt^2$ we the have then $\alpha = -2\,(R(0)^2/c^2)\,g$. Inserting c = 3,0 x 10^8 m/s, g = - 9,8 m/s^2 and R(0) = 6.4 x 10^6 meter we then find that α = 0,89 cm. We see that α << R(0) and so the approximation used is valid.

APPENDIX 2: RELATION BETWEEN THE CANONICAL AND THE SCHWARZSCHILD COORDINATES

The Schwarzschild coordinates t and φ are constructed identical to the canonical coordinates. Comparing the Schwarzschild metric : $ds^2= [1- (\alpha/R)]\,dt^2 - 1/c^2\,(1/[1-\alpha/R])\,dR^2 - 1/c^2\,R^2\,d\psi^2 - 1/c^2\,R^2\sin\psi^2\,d\varphi^2$ with the canonical metric $ds2 = f\,dt^2 - 1/c^2\,w\,dr^2 - 1/c^2\,w\,dz^2 - 1/c^2\,r^2/f\,d\varphi^2$ gives $f = 1 - \alpha/R$ and $r^2/f = R^2\sin\psi^2$ and so $r = \surd(R^2 - \alpha R)\sin\psi$. From this $dr = (\partial r/\partial R)\,dR + (\partial r/\partial psi)\,d\psi = [(R - \alpha/2)/\surd(R^2 - \alpha R)]\sin\psi\,dR + \surd(R^2 - \alpha R)\cos\psi\,d\psi$.

See figure 7.

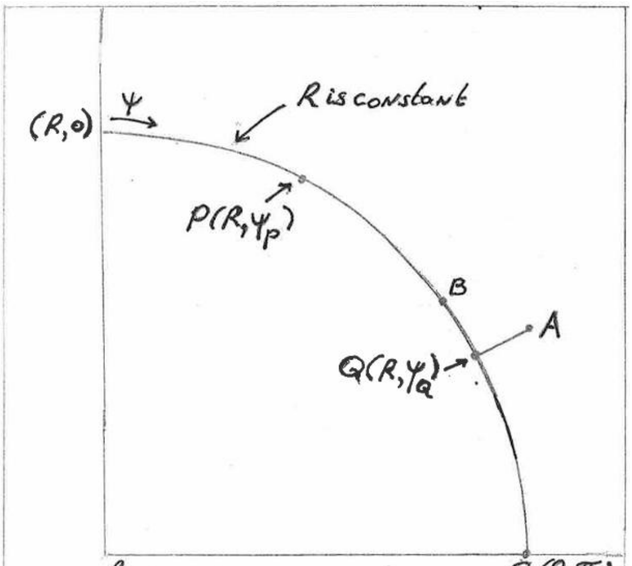


figure7

At Q (R, ψ_Q) we position the rectangular with arms of 1 meter tangential to the large circle with coordinate R in the plane φ = 0.For A then the Schwarzschild-coordinates are (R+dR, ψ_Q) with $dR = \surd(1-\alpha R)\,dh = \surd(1-\alpha/R)$.

Then $dr = [(R - \alpha/2)/\surd(R2 - \alpha R)]\sin\psi\,\surd(1-\alpha/R) = [(R-\alpha 2)/R]\sin\psi$. We see that dr depends on ψ.

The rectangular determines by the construction of z that B has the canonical coordinate z_B that is equal to $z_Q + dz = z_Q + dr = z_Q + [(R-\alpha/2)/R]\sin\psi$. We see that by construction dz also depends on ψ.

We found earlier R = 1/dψ(1meter) or 1/R = dψ(1meter) and so we find for dz that dz = $[(R-\alpha/2)/R]\sin\psi = (R-\alpha/2)\sin\psi\,d\psi$(1meter).

By integrating dz along the circle from the point S(R, π/2) on the equator to P (R, ψ_P) we find $z_P = (R - \alpha 2)\cos\psi_P$.

APPENDIX 3: DETERMINATION OF THE VELOCITY OF THE "MOVING" CABIN

We found for P* of the "moving" cabin $d\varphi^*/dt = qc^2 / l$. For $l = (r^2 - q^2c^2) / f$ we then have $d\varphi^* = [qc^2f /(r^2 - q^2c^2)]\, dt$.

From $ds^2 = f\, dt^2 - 1/c^2\, w\, dr^2 - 1/c^2\, w\, dz^2 - 1/c^2\, [(r^2-q^2c^2)/f]\, d\varphi^2 + 2\, q\, d\varphi\, dt$ we have for the movement of P* for $dr = 0$ and $dz = 0$: $ds^2 = f\, dt^2 - (1/c^2)\, [(r^2 - q^2c^2)/ f]\, d\varphi^{*2} + 2q d\varphi^*\, dt = f\, dt^2 - (1/c^2)\, [(r^2 - q^2c^2)/ f]\, [qc^2f/(r^2 - q^2c^2)]^2\, dt^2 + 2q\, [qc^2f/(r^2 - q^2c^2)]\, dt^2$ Rearranging gives $ds^2 =[1 / [1 - (q^2c^2/r^2)]\, f\, dt^2$ ... (1)

With respect to the Lorentzian coordinates (T', x') of the "moving" cabin we have for the movement of P* that $dx' = 0$ and so from $ds^2 = dT'^2 - 1/c^2\, dx'^2$ we have $ds^2 = dT'^2$ ... (2)

With respect to the Lorentzian coordinates (T, x) of the cabin at rest at $\varphi = 0$ we have for that movement $ds^2 = dT^2 - 1/c^2\, dx^2$. In this the value of dx is the distance in meters traveled by P* with respect to the cabin at rest. That distance must correspond to $d\varphi^*= [qc^2f/(r^2 - q^2c^2)]\, dt$.

From 5.7 we know $(q^2c^2 + lf)\, d\varphi(1meter)^2 = f$ and so to a rod of 1 meter corresponds $d\varphi(1meter) = \surd f)/\surd(q^2c^2 + lf) = \surd f/ r$. From this we see that dx must have the value:

$$dx = dfi^*/dfi(1meter)$$

$$= [qc^2f/(r^2 - q^2c^2)]\, dt\, /\, (\surd f/ r)$$

$$= [rqc^2 /(r^2 - q^2c^2)]\, (\surd f)\, dt$$

From $ds^2 = dT^2 - 1/c^2\, dx^2$ we then find for the movement of P* with respect to the cabin at rest:

$$ds2 = dT^2 - 1/c^2\, [(r^2q^2c^4) /(r^2 - q^2c^2)^2]\, f\, dt2 \text{ ... (3)}$$

From (1) and (3) then $[1 / (1 - (q^2c^2/r^2)]\, f\, dt^2 = dT2 - [(r^2q^2c^2) /(r^2 - q^2c^2)^2]\, f\, dt2$. After rearranging this gives $dT = [1 / (1 - q^2c^2/r^2)]\, (\surd f)\, dt$

Above we found $dx = [rqc^2)/(r^2 - q^2c^2]\, (\surd f)\, dt$ and so the velocity of P* with respect to the cabin at rest equals :

$$v = dx/dT$$

$= [rqc^2/(r^2 - q^2c^2)]\, (\surd f)\, dt / \{[1 / (1 - q^2c^2/r^2)]\, (\surd f)\, dt\}$ After rearranging this gives $v = qc^2/r$.

From (1) and (2) we have :

$[1 / (1 - (q^2c^2/r^2)]\, f\, dt^2 = dT'^2$ and so $dt^2 = (1 - q^2c^2/r^2)\, dT'^2 /f$. From (2) and (3) we have $dT'^2 = dT^2 - [(r^2q^2c^2) /(r^2 - q^2c^2)^2]\, f\, dt^2$.

By eliminating dt from the last equations we find $dT' = \surd(1-q2c2/r2)\, dT$. This is the familiar relation from SR between T' of a moving cabin and T of a cabin at rest for $q^2c^2/r^2 = v^2/c^2$, thus for $v = qc^2/ r$. This result corroborates the earlier result $v = qc^2/ r$.

APPENDIX 4: THE PATHS OF RADIAL PULSES OF LIGHT AND TEST-BODIES FALLING FROM REST

Pulses of light and test-bodies are assumed in GR to follow geodesic lines [17]. We choose a point P in the equatorial plane and in the plane $\varphi = 0$. From P we send a pulse of light in the equatorial plane and in the direction of the plane φ= 0. So for that pulse of light $(dz/dt)_P = 0$ and $(d\varphi/dt)_P = 0$. Let μ be a parameter of the path. The path is then determined by four equations of which two are given by:

$$d/d\mu\,(g_{33}\, d\varphi/d\mu + g_{34}\, dt/d\mu) = 0 \quad d/d\mu\,(g_{44}\, dt/d\mu + g_{43}\, d\varphi/d\mu) = 0$$

From these we have for the path of the pulse of light, remembering $g_{34} = g_{43}$:

$g_{33}\, d\varphi/d\mu + g_{34}\, dt/d\mu = A$ and $g_{44}\, dt/d\mu + g_{34}\, d\varphi/d\mu = B$ in which A and B are constants of the path. Solving for dφ/dμ and dt/dψi gives :

$$d\varphi/d\mu = (A\, g_{44} - B\, g_{34}) / (g_{33}\, g_{44} - g_{34}^2) \quad dt/d\mu = (B\, g_{33} - A\, g_{34}) / (g_{33}\, g_{44} - g_{34}^2)$$

From these we have $d\varphi/dt = (A\, g_{44} - B\, g_{34}) / (B\, g_{33} - A\, g_{34})$

To regain $(d\varphi/dt)_P = 0$ we choose the constants $A = (g_{34})_P$ and $B = (g_{44})_P$. For some other point Q of the path we then have:

$$(d\varphi/dt)_Q = [A\, (g_{44})_Q - B\, (g_{34})_Q] / [B\, (g_{33})_Q - A\, (g_{34})_Q)].$$

This value is generally different from 0 and consequently the pulse of light deviates from the plane φ = 0. In the same way it can be shown that a test-body falling from rest will deviate from the plane φ = 0.