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Holomorphy in Pseudo-Euclidean Spaces and the Classic Electromagnetic Theory

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ABSTRACT

A new concept of holomorphy in pseudo-Euclidean spaces is briefly presented. The set of extended Cauchy-Riemannn differential equations, which are verified by the holomorphic functions, is obtained. A form of the general pseudo-rotation matrix was developed. The generalized d'Alembert- operator and extended Poisson's equations are defined. Applying these results to the relativistic space-time, the charge conservation and general Maxwell equations are derived.

1 Introduction

In a paper [1], published in 1981, Salingaros proposed an extension of the Cauchy-Riemann equations of holomorphy to fields in higher-dimensional spaces. He formulated the theory of holomorphic fields by using Clifford algebras [2]. In the Minkowski space-time he found out that the equations of holomorphy are identical with the Maxwell equations in vacuum.

In the present article we introduce a different definition of monogenity/holomorphy applied to vector functions in a pseudo-Euclidean space. This enables us to obtain a set of equations, which applied to the Minkowski space-time, lead to general Maxwell equations and to the charge conservation law. All physical quantities involved in the ongoing presentation are expressed in geometric units [3], i.e. meters.

2 Preliminary

2.1 Pseudo-rotation and its transformation matrix

Let us consider a Riemannian n-dimensional space with the metric [4]:

$$d^2 s_x = \sum_{i,k=1}^{n} g_{ik} dx_i dx_k \tag{1.1}$$

If the g_{ik} coefficients are constant, then the space is called pseudo-Euclidean and the coordinate system is rectilinear. Using linear transformations we obtain a new coordinates system:

$$x_i^{'} = x_i^{'}(x_1, x_2, \ldots\ldots\ldots x_n) \tag{1.2}$$

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Below we have the expression of the Jacobian matrix of this transformation.

$$J = \left[\frac{\partial x'}{\partial x_1}, \frac{\partial x'}{\partial x_2}, ... \frac{\partial x'}{\partial x_n}\right] = \begin{pmatrix} \frac{\partial x'_1}{\partial x_1} & \frac{\partial x'_1}{\partial x_2} & . & \frac{\partial x'_1}{\partial x_n} \\ \frac{\partial x'_2}{\partial x_1} & \frac{\partial x'_2}{\partial x_2} & . & \frac{\partial x'_2}{\partial x_n} \\ . & . & . & . \\ \frac{\partial x'_n}{\partial x_1} & \frac{\partial x'_n}{\partial x_2} & . & \frac{\partial x'_n}{\partial x_n} \end{pmatrix} \tag{1.3}$$

If the value of ds_x remains unmodified, then this transformation will be generally named a ***pseudo-rotation***. The transformation becomes **pure rotation** in the case of Euclidean spaces.

$$\begin{gathered} d^2 s_x = \sum_{i,k=1}^{n} g_{ik} dx_i dx_k = \sum_{i,k}^{n} g'_{ik} dx'_i dx'_k = d^2 s_{x'} \\ \frac{d^2 s_{x'}}{d^2 s_x} = 1 \end{gathered} \tag{1.4}$$

We will consider further only transformations where $g_{ik} = g'_{ik}$.

Developing the differentials in the right side of the first equation (1.4) and identifying, it obtains the following **important relationship**:

$$g_{jp} = \sum_{i,k=1}^{n} g_{ik} \frac{\partial x'_i}{\partial x_j} \frac{\partial x'_k}{\partial x_p} \tag{1.5}$$

2.2 Holomorphy in n-dimensional spaces

If it considered a vector field $\mathbf{f} = (f_1, f_2, ... f_n)$, defined on an n-dimensional space, then its Jacobi matrix is as follows:

$$J = \left[\frac{\partial \mathbf{f}}{\partial x_1}, \frac{\partial \mathbf{f}}{\partial x_2}, ... \frac{\partial \mathbf{f}}{\partial x_n}\right] = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & . & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & . & \frac{\partial f_2}{\partial x_n} \\ . & . & . & . \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & . & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \tag{1.6}$$

The differential of this field function can be written as:

$$d\mathbf{f} = (df_1, df_2, ... df_n) \tag{1.7}$$

Using the metric definition (1.1) we may write the norm of this differential expression $d^2 s_f = \sum_{i,k=1}^{n} g_{ik} df_i df_k$

Definition

A vector field $\mathbf{f} = [f_1(x), f_2(x), \ldots\ldots\ldots f_n(x)]$, where $f_i(x) = f_i(x_1, x_2, \ldots. x_n)$, is *said to be monogenic at point x of the space if the ratio*:

$$\frac{d^2 s_f}{d^2 s_x} = \frac{\sum_{i,k=1}^{n} g_{ik} df_i df_k}{\sum_{i,k=1}^{n} g_{ik} dx_i dx_k} = \pm\Omega^2(x), \tag{1.8}$$

<u>exists and is unique</u> at this point. If a vector field f is monogenic in all the points belonging to a set D in space, then f is holomorphic in the set D.

For further developments we consider only the sign + in the right side of the equation (1.8). The uniqueness condition (1.8) requires that:

$$g_{jp} = \sum_{i,k=1}^{n} g_{ik} \left(\frac{1}{\Omega}\frac{\partial f_i}{\partial x_j}\right)\left(\frac{1}{\Omega}\frac{\partial f_k}{\partial x_p}\right) \tag{1.9}$$

Comparing with (1.5) and (1.3) it obtains the following set of equations:

$$\frac{1}{\Omega}\frac{\partial f_i}{\partial x_j} = \frac{\partial x_i'}{\partial x_j}, \tag{1.10}$$

where $i, j = 1, 2, \ldots\ldots\ldots n.$

Equations (1.10) can be considered as **the extension of the Cauchy-Riemann equations** to an n-dimensional space. Further it will be considered only pseudo-Euclidean spaces where:

$$g_{ik} = g'_{ki} = 0$$
$$g_{ii} = g'_{ii} = c_i$$

More than that we may state, for simplicity, that c_i is either 1 or -1.

2.3 Pseudo-rotation matrices and associated Cauchy-Riemann equations

Let us consider a $n \times n$ matrix M, which performs the coordinate transformation $\mathbf{x} \to \mathbf{x'}$ in an n-dimensional space:

$$M = \begin{pmatrix} a_{11} & a_{12} & . & a_{1n} \\ a_{21} & a_{22} & . & a_{2n} \\ . & . & . & . \\ a_{n1} & a_{n2} & . & a_{nn} \end{pmatrix} \tag{1.11}$$

M is a pseudo-rotation matrix if and only if its columns verify the equation (1.5). This is the necessary and sufficient condition for M to be a called a pseudo-rotation matrix.

a. As a first example we consider one of the rotation matrices in the two dimensional Euclidean space, where $c_1 = c_2 = 1$.

$$M = \begin{bmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{bmatrix}$$

Using (1.10) it obtains the original Cauchy-Riemann equations, known from complex analysis

$$\begin{aligned} \frac{\partial f_1}{\partial x_1} &= \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_1}{\partial x_2} &= -\frac{\partial f_2}{\partial x_1} \end{aligned} \tag{1.12}$$

Considering the definition (1.8), the conditions (1.12) are not unique. An alternate valid form of a rotation matrix in this space could be:

$$M = \begin{bmatrix} \cos\gamma & \sin\gamma \\ \sin\gamma & -\cos\gamma \end{bmatrix}$$

Consequently the extended Cauchy-Riemann equations look differently:

$$\begin{aligned} \frac{\partial f_1}{\partial x_1} &= -\frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_1}{\partial x_2} &= \frac{\partial f_2}{\partial x_1} \end{aligned} \tag{1.13}$$

In both cases the functions are harmonic and satisfy the Laplace's equations:

$$\frac{\partial^2 f_1}{\partial x_1^2} + \frac{\partial^2 f_1}{\partial x_2^2} = 0$$

$$\frac{\partial^2 f_2}{\partial x_1^2} + \frac{\partial^2 f_2}{\partial x_2^2} = 0$$

b. A pseudo-rotation matrix in a two-dimensional pseudo-Euclidean space, where $c_1 = -1$ and $c_2 = 1$, can have the following form :

$$M = \begin{bmatrix} \cosh\gamma & -\sinh\gamma \\ -\sin\gamma & \cosh\gamma \end{bmatrix}$$

M is the matrix of the Lorenz transformations in the two dimensional **space**(x_2)-**time**(x_1), and the corresponding Cauchy-Riemann equations system is shown below:

$$\begin{aligned} \frac{\partial f_1}{\partial x_1} &= \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_1}{\partial x_2} &= \frac{\partial f_2}{\partial x_1} \end{aligned} \tag{1.14}$$

The above functions satisfy the wave equation in one space dimension.

$$\begin{aligned} \frac{\partial^2 f_1}{\partial x_1^2} &= \frac{\partial^2 f_1}{\partial x_2^2} \\ \frac{\partial^2 f_2}{\partial x_1^2} &= \frac{\partial^2 f_2}{\partial x_2^2} \end{aligned} \tag{1.15}$$

2.4 Generalized d'Alembert operator and extended Poisson's equations

As we have seen in the previous paragraph, the pseudo-rotation matrices on the same space are not identical and their freedom degree grows with the dimensions number, n. This implies that for the same type of space there are different Cauchy-Riemann equations which provide necessary conditions for a vector field **f** to be holomorphic. One of possible forms of a general pseudo-rotation matrix may be as follows:

$$M = \begin{pmatrix} a_{11} & a_{12} & . & a_{1n} \\ a_{21} & a_{22} & . & a_{2n} \\ . & . & . & . \\ a_{n1} & a_{n2} & , & a_{nn} \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 & . & \alpha_n \\ \alpha_2 & c_2 - \dfrac{\alpha_2{}^2}{c_1 - \alpha_1} & . & -\dfrac{\alpha_2\alpha_n}{c_1 - \alpha_1} \\ . & . & . & . \\ \alpha_n & -\dfrac{\alpha_2\alpha_3}{c_1 - \alpha_1} & . & c_n - \dfrac{\alpha_n{}^2}{c_1 - \alpha_1} \end{pmatrix} \tag{1.16}$$

It can be verified that all columns of M satisfy the equation (1.5), and also that:

$$a_{ik} = a_{ki} \tag{1.17}$$

Now let us process the elements of the diagonal which starts with α_1, in according with the following relationship

$$\begin{aligned} S_{diagonal} &= \sum_{i=1}^{n} c_i a_{ii} \\ S_{diagonal} &= c_1\alpha_1 + c_2(c_2 - \frac{\alpha_2{}^2}{c_1 - \alpha_1}) + c_3(c_3 - \frac{\alpha_3{}^2}{c_1 - \alpha_1})\ldots\ldots + c_n(c_n - \frac{\alpha_n{}^2}{c_1 - \alpha_1}) \\ S_{diagonal} &= n - 1 + \frac{\alpha_1 - c_1\alpha_1{}^2 - c_2\alpha_2{}^2 \ldots\ldots - c_n\alpha_n{}^2}{c_1 - 1} = n - 1 + \frac{\alpha_1 - c_1}{c_1 - \alpha_1} = n - 2 \end{aligned} \tag{1.18}$$

For the four-dimensional Minkowski space-time, c_1=-1, c_2=c_3=c_4=1, the corresponding matrix becomes:

$$M = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_2 & 1+\frac{\alpha_2{}^2}{1+\alpha_1} & \frac{\alpha_2\alpha_3}{1+\alpha_1} & \frac{\alpha_2\alpha_4}{1+\alpha_1} \\ \alpha_3 & \frac{\alpha_3\alpha_2}{1+\alpha_1} & 1+\frac{\alpha_3{}^2}{1+\alpha_1} & \frac{\alpha_3\alpha_4}{1+\alpha_1} \\ \alpha_4 & \frac{\alpha_4\alpha_2}{1+\alpha_1} & \frac{\alpha_4\alpha_3}{1+\alpha_1} & 1+\frac{\alpha_4{}^2}{1+\alpha_1} \end{bmatrix}$$

Taking the appropriate substitutions and computing, it obtains the matrix of the standard Lorenz transformations as you can see in the reference [5] (equations 1.17 and the corresponding matrix).

Using the equations (1.17), (1.18) and (1.10) it obtains the following relationships:

$$\begin{gathered} \frac{\partial f_i}{\partial x_k} = \frac{\partial f_k}{\partial x_i} \\ \sum_{i=1}^{n} c_i \frac{\partial f_i}{\partial x_i} = (n-2)\Omega = \Lambda \end{gathered} \tag{1.19}$$

Processing (1.19) it arrives to the following expression:

$$\sum_{i=1}^{n} c_i \frac{\partial^2 f_k}{\partial x_i^2} = \partial^2 f_k = \frac{\partial \Lambda}{\partial x_k} \tag{1.20}$$

Further the symbol ∂^2 will be named **d'Alembert operator** of the n-dimensional pseudo-Euclidean space.

The equation (1.20) is denominated **the extended Poisson's equation** in the same space.

For n=2 it obtains the Laplace equations and the wave equations, previously developed in the paragraph 1.2.

For the Minkowski space-time, with the signature (-1, 1, 1, 1), it will be used further the standard denomination of the coordinates, i.e. t, x_1, x_2, x_3 .

The corresponding vector field has the following expression:

$$\mathbf{f} = (T, X_1, X_2, X_3) \tag{1.21}$$

Using equations (1.20) it obtains:

$$\begin{gathered} \partial^2 T = \frac{\partial \Lambda}{\partial t} \\ \partial^2 X_i = \frac{\partial \Lambda}{\partial x_i} \end{gathered} \tag{1.22}$$

The symbol $\Lambda = 2\Omega$ represents a function at the point $P(t, x_1, x_2, x_3)$.

X_1, X_2, X_3 are the components of the following vector in the three-dimensional Euclidean space:

$$\mathbf{X} = \sum_{i=1}^{3} X_i \mathbf{e_i} \tag{1.23}$$

where $\mathbf{e_i}$ are unit vectors along the Cartesian axes of this space.

The last three equations of the system (1.22) will be packed together, and so the system takes the following format:

$$\begin{aligned} &\partial^2 \mathbf{X} = \nabla \Lambda \\ &\partial^2 \mathrm{T} = \frac{\partial \Lambda}{\partial \mathrm{t}} \end{aligned} \tag{1.24}$$

In the system above it was used the "del" operator in the three dimensional Euclidean space.

If we consider the pair of inhomogeneous wave equations[6] for electromagnetic potentials, then the system (1.24) shows a perfect similarity. We are very tempted to identify T with the electro-magnetic scalar potential and the vector **X** with the vector potential, but it does not work because the first equation of the system (1.19) requires that the curl-operator or rotation-operator of X must be zero. This is not generally valid for a real electro-magnetic vector-potential.

3 Classical Electrodynamics and Maxwell equations

3.1 Alternative Cauchy-Riemann equations in Space-Time.

There is a class of matrices in the Minkowski space-time which fulfills the following relations:

$$\begin{aligned} &\sum_{i=1}^{4} \mathrm{a}_{ii} = 0 \\ &\mathrm{a}_{12} + \mathrm{a}_{21} = \mathrm{c}_2 \\ &\mathrm{a}_{13} + \mathrm{a}_{31} = \mathrm{c}_3 \\ &\mathrm{a}_{14} + \mathrm{a}_{41} = \mathrm{c}_4 \end{aligned} \tag{2.1}$$

Replacing by partial derivatives, in according with equations (1.10) and using the usual coordinate's notation for Minkowski space-time, it obtains:

$$\begin{aligned} &\frac{\partial \mathrm{T}}{\partial \mathrm{t}} + \nabla \bullet \mathbf{X} = 0 \\ &\nabla \mathrm{T} + \frac{\partial \mathbf{X}}{\partial \mathrm{t}} = \frac{\partial \mathbf{C}}{\partial \mathrm{t}} \end{aligned} \tag{2.2}$$

We also can obtain the equations system (2.2) considering a vector field, $\mathbf{f} = (\mathrm{T}, \mathrm{X'}_1, \mathrm{X'}_2, \mathrm{X'}_3)$ in Minkowski space-time which fulfils the conditions (1.19). If $\mathbf{X}'$ represents the corresponding vector in the three dimensional Euclidean space, then respective equations become:

$$-\frac{\partial T}{\partial t} + \nabla \bullet \mathbf{X}' = \Lambda = \nabla \bullet \mathbf{C}$$
$$\nabla T - \frac{\partial \mathbf{X}'}{\partial t} = 0 \tag{2.3}$$

Now let us make the substitution: $\mathbf{X} = \mathbf{C} - \mathbf{X}'$

Replacing in (2.3) we arrive again at the system (2.2). The second equation of the system (2.2) implies that:

$$\nabla \times \mathbf{X} = \nabla \times \mathbf{C} \tag{2.4}$$

Now let us convert to the SI system (see reference [3]) identifying the scalar part with the scalar potential and the vector part with the vector potential. Further we will use the symbols shown in the reference [6] for these potentials. It obtains the following system of equations:

$$\frac{\partial \Phi}{\partial t} + c^2 \nabla \mathbf{A} = 0$$
$$\nabla \Phi + \frac{\partial \mathbf{A}}{\partial t} = \frac{\partial \mathbf{C}}{\partial t} \tag{2.5}$$

Where c is the velocity of light in SI units system. Processing further we get:

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = \nabla \bullet (\frac{\partial \mathbf{C}}{\partial t})$$
$$\nabla(\nabla \bullet \mathbf{A}) - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{1}{c^2}\frac{\partial^2 \mathbf{C}}{\partial t^2} \tag{2.6}$$

But:

$\nabla(\nabla \bullet \mathbf{A}) = \nabla^2 \mathbf{A} + \nabla \times (\nabla \times \mathbf{A})$, and finally the system (2.6) becomes:

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{1}{c^2}\frac{\partial^2 \mathbf{C}}{\partial t^2} - \nabla \times (\nabla \times \mathbf{A}) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{C}}{\partial t^2} - \nabla \times (\nabla \times \mathbf{C})$$
$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = \nabla \bullet (\frac{\partial \mathbf{C}}{\partial t}) \tag{2.7}$$

3.2 Charge conservation and Maxwell's Equations

Identifying (2.7) with inhomogeneous wave equations presented by Feynman [6] we find the expressions for charge and current density.

$$\rho = -\varepsilon_0 \nabla \bullet (\frac{\partial \mathbf{C}}{\partial t})$$
$$\mathbf{j} = \varepsilon_0 \left[\frac{\partial^2 \mathbf{C}}{\partial t^2} + c^2 \nabla \times (\nabla \times \mathbf{C}) \right] \tag{2.8}$$

Processing the equations (2.8) it obtains immediately the well known **equation of charge conservation**:

$$\nabla \bullet \mathbf{j} = -\frac{\partial \rho}{\partial t} \tag{2.9}$$

Using the equations (2.5 we can write further the expressions of electric field intensity $\boldsymbol{E}$ and the magnetic induction **B:**

$$\begin{aligned} \boldsymbol{E} &= -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} = -\frac{\partial \mathbf{C}}{\partial t}, \\ \boldsymbol{B} &= \nabla \times \mathbf{A} = \nabla \times \mathbf{C} \end{aligned} \tag{2.10}$$

a. ***The first Maxwell's law***

Taking the divergence of **E** and using the first equation of the system (2.8) it obtains the first Maxwell's law:

$$\nabla \bullet \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{2.11}$$

b. ***The second Maxwell's law***

Taking the curl of **E** and comparing with the expression of **B** we get immediately the second law:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.12}$$

c. ***The third Maxwell's law***

The third Maxwell's equation is evident because ***B*** is curl $\mathbf{C}$:

$$\nabla \bullet \mathbf{B} = 0 \tag{2.13}$$

d. ***The fourth Maxwell's law***

Processing the second equation of the system (2.8) and taking into consideration (2.10) we found finally the fourth law:

$$c^2 \nabla \times \mathbf{B} = \frac{\mathbf{j}}{\varepsilon_0} + \frac{\partial \mathbf{E}}{\partial t} \tag{2.14}$$

4 Conclusion

The Lorenz Transformation equations have been initially derived searching for a transformation, which leaves the Maxwell's equations invariant [7]. The immediate consequence of the Lorenz transformations is Einstein's Special Relativity. In the present contribution we found that any four-vector, which is holomorphic in a domain of the space-time, must verify the system (1.10). As a first application we rediscovered the law of charge conservation and all four Maxwell's equations.

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