

Dynamic of Bacterial and Phage Populations Maintained in Chemostat System with Dilution Control Parameter

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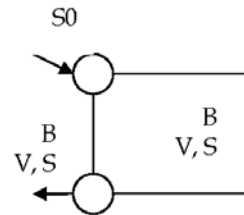
ABSTRACT — In this article, we discuss the crucial subject of bacteria and phage populations dynamic in chemostat system with dilution control parameter. A chemostat (from chemical environment is static) is a bioreactor to which fresh medium is continuously added, while culture liquid is continuously removed to keep the culture volume constant. By changing the rate with which medium is added to the bioreactor the growth rate of the microorganisms can be controlled. In chemostat model, the control parameters usually are the washout rate (dilution rate D) and input nutrient concentration $S^{(0)}$. We consider that $S^{(0)}$ is fixed and D is the only control parameter. The outputs considered in chemostat models are functions: $\xi : R^3 \rightarrow R^k$ ($k \leq 3$) of variables $S(t), V(t)$ and $B(t)$. The dynamical behavior of chemostat system with control tools is inspected and stability analysis is done. The chemostat system with dilution control parameter can be represented as differential equations, depending on variable and parameters. First the system is analyzed for zero talent period ($\tau = 0$) and secondly for positive value of talent period ($\tau > 0$).

Index Terms: Chemostat, Bioreactor, Dilution rate, Fixed point, Stability, Talent period, Delay Differential Equation (DDE).

I. INTRODUCTION

In this article, we discuss the crucial subject of bacteria and phage populations dynamic in chemostat system with dilution control parameter. In Chemostat model, the control variables employed are the washout rate (dilution rate D) and input nutrient concentration $S^{(0)}$. First we consider that $S^{(0)}$ is fixed and D will be the only control variable. The outputs usually considered in Chemostat models are functions: $\xi : R^3 \rightarrow R^k$ ($k \leq 3$) of variables $S(t), V(t)$ and $B(t)$. We consider that dilution rate D is the feedback control variable of the system and define the feedback control law $D : R_+^3 \rightarrow R_+$ by $D(B, V, S) = \xi(B, V, S)$ where $\xi : R_+ \rightarrow R_+$ is a continuously differentiable. $D[1/sec]$, $B[1/ml]$, $V[1/ml]$, $S[1/ml]$. We define washout rate balance parameters $k_i[ml/sec]$. Dilution rate function: $D = \xi(B, V, S)$ $D = \xi(B, V, S) = Bk_1 + Vk_2 + Sk_3$ [1] [2] [3].

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$$D = \xi(B, V, S) = Bk_1 + Vk_2 + Sk_3$$

Fig. 1. Chemostat system with dilution control parameter.

$$D\left[\frac{1}{sec}\right] = B\left[\frac{1}{ml}\right]k_1\left[\frac{ml}{sec}\right] + V\left[\frac{1}{ml}\right]k_2\left[\frac{ml}{sec}\right] + S\left[\frac{1}{ml}\right]k_3\left[\frac{ml}{sec}\right] \quad (1)$$

Terminology:

V - Phage concentration in the vessel.

B - Bacteria concentration in the vessel.

$S^{(0)}$ - Input concentration of the limiting nutrient.

S - Current concentration of the nutrient in the vessel.

System chemostat (1compartment gradostat) equations:

$$\frac{dV(t)}{dt} = -V(t) - \gamma V(t)B(t) + \beta \gamma V(t - \tau)B(t - \tau) - V(t)D \quad (2)$$

$$\frac{dB(t)}{dt} = \psi f(S)B(t) - \gamma V(t)B(t) - B(t)D \quad (3)$$

$$\frac{dS(t)}{dt} = [S^{(0)} - S(t)]D - \psi f(S)B(t)/\delta \quad (4)$$

$$D = \xi(B, V, S) = Bk_1 + Vk_2 + Sk_3 \quad (5)$$

φ - Decay rate of phage in the vessel.

β - Average number of free phage released at lysing in the vessel.

ψ - Growth rate of bacteria in the vessel.

γ - Interaction rate between bacteria and phage in the vessel.

τ - Talent period. D - Washout rate.

$S^{(0)}$ - Input concentration of the limiting nutrient.

δ - Conversion factor in the vessel.

k_1, k_2, k_3 - Washout rate balance parameters.

$S^{(0)} > 0, B(0) > 0, V(0) > 0$. Chemostat washout term $[S^{(0)} - S(t)]D$. For the case of simplicity we assume no delay $\tau = 0$.

$$\begin{aligned} \frac{dV(t)}{dt} &= -\varphi V(t) + \gamma V(t)B(t)(\beta - 1) \\ &\quad - V(t)[B(t)k_1 + V(t)k_2 + S(t)k_3] \\ \frac{dB(t)}{dt} &= \psi f(S)B(t) - \gamma V(t)B(t) \\ &\quad - B(t)[B(t)k_1 + V(t)k_2 + S(t)k_3] \\ \frac{dS(t)}{dt} &= [S^{(0)} - S(t)][B(t)k_1 + V(t)k_2 + S(t)k_3] \\ &\quad - \psi f(S)B(t)/\delta \end{aligned} \quad (6)$$

$$V = V(t); B = B(t); S = S(t) \quad (7)$$

$$\begin{aligned} \frac{dV}{dt} &= -\varphi V + \gamma VB(\beta - 1) - V[Bk_1 + Vk_2 + Sk_3] \\ \frac{dB}{dt} &= \psi f(S)B - \gamma VB - B[Bk_1 + Vk_2 + Sk_3] \\ \frac{dS}{dt} &= [S^{(0)} - S][Bk_1 + Vk_2 + Sk_3] - \psi f(S)B/\delta \end{aligned} \quad (8)$$

Scaling of the above equation: All resources densities in term of $S^{(0)}$ and get

$$\frac{S}{S^{(0)}} \rightarrow S; \frac{V}{S^{(0)}\beta\delta} \rightarrow V; \frac{B}{S^{(0)}\delta} \rightarrow B; a \rightarrow \frac{a}{S^{(0)}} \quad (9)$$

$t \rightarrow tD$. Dimensional parameters have the following meaning:

$$\psi \rightarrow \frac{\psi}{D}; \gamma \rightarrow \frac{\gamma\beta S^{(0)}\delta}{D}; k_1 \rightarrow k_1\delta S^{(0)} \quad (10)$$

$$k_2 \rightarrow k_2 S^{(0)}\beta\delta; k_3 \rightarrow S^{(0)}k_3; S^{(0)} \geq 0 \quad (11)$$

$$B(0) > 0; \frac{S}{S^{(0)}} \rightarrow S \quad (12)$$

Input concentration of the limiting nutrient $S^{(0)}$ term has units of concentration. $D[1/sec]$ has units of reciprocal time. Let scale the nutrient equation:

$$\frac{dS}{dt} = [S^{(0)} - S][Bk_1 + Vk_2 + Sk_3] - \psi f(S)B/\delta \quad (13)$$

Divide the two equation sides by $S^{(0)}$ term:

$$\frac{dS}{dt} \frac{1}{S^{(0)}} = [1 - \frac{S}{S^{(0)}}][\frac{B}{S^{(0)}\delta}k_1\delta S^{(0)} + \frac{V}{S^{(0)}\beta\delta}k_2 S^{(0)}\beta\delta + \frac{S}{S^{(0)}}S^{(0)}k_3] - \psi f(S)B/[S^{(0)}\delta] \quad (14)$$

Measuring $S, a, B/\delta$ in units of $S^{(0)}$, and time in units of D^{-1}

$$\frac{B[\frac{1}{ml}]}{S^{(0)}[\frac{1}{ml}]\delta}k_1[\frac{ml}{sec}]\delta S^{(0)}[\frac{1}{ml}] \rightarrow [\frac{1}{sec}] \quad (15)$$

$$\frac{V[\frac{1}{ml}]}{S^{(0)}[\frac{1}{ml}]\beta\delta}k_2[\frac{ml}{sec}]S^{(0)}[\frac{1}{ml}]\beta\delta \rightarrow [\frac{1}{sec}] \quad (16)$$

The scales nutrient equation is:

$$\frac{dS}{dt} = (1 - S)(Bk_1 + Vk_2 + Sk_3) - \psi f(S)B \quad (17)$$

Let take the phages differential equation:

$$\frac{V}{S^{(0)}\beta\delta} \rightarrow V; \frac{\varphi[\frac{1}{sec}]}{D[\frac{1}{sec}]} \rightarrow \varphi \quad (18)$$

$$\frac{dV}{dt} = -\varphi V + \gamma VB(\beta - 1) - V[Bk_1 + Vk_2 + Sk_3] \quad (19)$$

Rewriting $V(t)$ phage density in terms of $S^{(0)}\beta\delta$

$$\begin{aligned} \frac{dV}{dt} \frac{1}{S^{(0)}\beta\delta} &= -\varphi V \frac{1}{S^{(0)}\beta\delta} + \gamma V \frac{1}{S^{(0)}\beta\delta} B(\beta - 1) \\ &\quad - V \frac{1}{S^{(0)}\beta\delta} [\frac{B}{S^{(0)}\delta}k_1 S^{(0)}\delta + V \frac{1}{S^{(0)}\beta\delta}k_2 S^{(0)}\beta\delta \\ &\quad + \frac{S}{S^{(0)}}k_3 S^{(0)}] \end{aligned} \quad (20)$$

$$\begin{aligned} \frac{dV}{dt} \frac{1}{S^{(0)}\beta\delta} &= -\varphi V \frac{1}{S^{(0)}\beta\delta} \\ &\quad + \gamma \frac{S^{(0)}\beta\delta}{D} V \frac{1}{S^{(0)}\beta\delta} B \frac{D}{S^{(0)}\beta\delta} (\beta - 1) \\ &\quad - V \frac{1}{S^{(0)}\beta\delta} [\frac{B}{S^{(0)}\delta}k_1 S^{(0)}\delta \\ &\quad + V \frac{1}{S^{(0)}\beta\delta}k_2 S^{(0)}\beta\delta + \frac{S}{S^{(0)}}k_3 S^{(0)}] \end{aligned} \quad (21)$$

$$\begin{aligned} \frac{dV}{dt} \frac{1}{S^{(0)}\beta\delta} &= -\varphi V \frac{1}{S^{(0)}\beta\delta} \\ &\quad + \gamma \frac{S^{(0)}\beta\delta}{D} V \frac{1}{S^{(0)}\beta\delta} B \frac{D}{S^{(0)}\beta\delta} (\beta - 1) \\ &\quad - V \frac{1}{S^{(0)}\beta\delta} [\frac{B}{S^{(0)}\delta}k_1 S^{(0)}\delta + V \frac{1}{S^{(0)}\beta\delta}k_2 S^{(0)}\beta\delta \\ &\quad + \frac{S}{S^{(0)}}k_3 S^{(0)}] \end{aligned} \quad (22)$$

$$V \frac{1}{S^{(0)}\beta\delta} \rightarrow V; B \frac{1}{S^{(0)}\delta} \rightarrow B; \gamma \frac{S^{(0)}\beta\delta}{D} \rightarrow \gamma \quad (23)$$

$$\begin{aligned} \varphi \rightarrow \frac{\varphi}{D}; \psi \rightarrow \frac{\psi}{D}; \gamma \rightarrow \frac{\gamma\beta S^{(0)}\delta}{D}; k_1 \rightarrow k_1\delta S^{(0)} \\ k_2 \rightarrow k_2 S^{(0)}\beta\delta; k_3 \rightarrow S^{(0)}k_3 \end{aligned} \quad (24)$$

$$\begin{aligned} S^{(0)} \geq 0; B(0) > 0; \frac{S}{S^{(0)}} \rightarrow S \\ \frac{a}{S^{(0)}} \rightarrow a; f(S) = \frac{V_{max}S/S^{(0)}}{a/S^{(0)} + S/S^{(0)}} \end{aligned} \quad (25)$$

$$\begin{aligned} \frac{S}{S^{(0)}} \rightarrow S; \frac{V}{S^{(0)}\beta\delta} \rightarrow V; \frac{B}{S^{(0)}\delta} \rightarrow B \\ a \rightarrow \frac{a}{S^{(0)}}; t \rightarrow tD \end{aligned} \quad (26)$$

$$\frac{dV}{dt} = -\varphi V + \gamma VB \frac{D}{\beta} (\beta - 1) - V[Bk_1 + Vk_2 + Sk_3] \quad (27)$$

Measuring $S, a, B/\delta$ in units of $S^{(0)}$, and time in units of D^{-1}

$$\frac{dV}{dt} \frac{1}{D} = -\varphi V + \gamma VB \frac{D}{\beta} \frac{1}{D} (\beta - 1) - V[Bk_1 + Vk_2 + Sk_3] \frac{1}{D} \quad (28)$$

$$\frac{dV}{dt} = -\varphi V + \gamma VB \frac{(\beta - 1)}{\beta} - V[Bk_1 + Vk_2 + Sk_3] \quad (29)$$

$$\frac{dV}{dt} = -\varphi V + \gamma VB(1 - \frac{1}{\beta}) - V[Bk_1 + Vk_2 + Sk_3] \quad (30)$$

Now let scale the Bacteria equations:

$$\frac{dB}{dt} = \psi f(S)B - \gamma VB - B[Bk_1 + Vk_2 + Sk_3] \quad (31)$$

We write the Bacteria densities in term of $S^{(0)}\delta$

$$\frac{B}{S^{(0)}\delta} \rightarrow B; \frac{V}{S^{(0)}\beta\delta} \rightarrow V; \gamma \rightarrow \frac{\gamma\beta S^{(0)}\delta}{D}$$

$$f(S) = \frac{V_{max}S/S^{(0)}}{a/S^{(0)}+S/S^{(0)}} \quad (32)$$

Multiply two equation sides by factor $\frac{1}{S^{(0)}\delta}$

$$\frac{dB}{dt} \frac{1}{S^{(0)}\delta} = \psi f(S)B \frac{1}{S^{(0)}\delta} - \gamma VB \frac{1}{S^{(0)}\delta}$$

$$- B \frac{1}{S^{(0)}\delta} [Bk_1 + Vk_2 + Sk_3] \quad (33)$$

$$\frac{dB}{dt} \frac{1}{S^{(0)}\delta} = \psi f(S)B \frac{1}{S^{(0)}\delta} - \gamma VB \frac{1}{S^{(0)}\delta}$$

$$- B \frac{1}{S^{(0)}\delta} [\frac{B}{S^{(0)}\delta} k_1 S^{(0)}\delta + V \frac{1}{S^{(0)}\beta\delta} k_2 S^{(0)}\beta\delta$$

$$+ \frac{S}{S^{(0)}} k_3 S^{(0)}]$$

$$\frac{dB}{dt} = \psi f(S)B - \gamma VDB - B[Bk_1 + Vk_2 + Sk_3] \quad (35)$$

$$\frac{dB}{dt} \frac{1}{D} = \psi \frac{1}{D} f(S)B - \gamma VB - B[Bk_1 + Vk_2 + Sk_3] \frac{1}{D} \quad (36)$$

$$\frac{dB}{dt} \frac{1}{D} \rightarrow \frac{dB}{dt}; \psi \frac{1}{D} \rightarrow \psi$$

$$[Bk_1 + Vk_2 + Sk_3] \frac{1}{D} \rightarrow [Bk_1 + Vk_2 + Sk_3] \quad (37)$$

$$\frac{dB}{dt} = \psi f(S)B - \gamma VB - B[Bk_1 + Vk_2 + Sk_3] \quad (38)$$

The Chemostat (One compartment Gradostat) scaled equations with control parameters are

$$\frac{dB}{dt} = \psi f(S)B - \gamma VB - B[Bk_1 + Vk_2 + Sk_3] \quad (39)$$

$$\frac{dV}{dt} = -\varphi V + \gamma VB(1 - \frac{1}{\beta}) - V[Bk_1 + Vk_2 + Sk_3] \quad (40)$$

$$\frac{dS}{dt} = (1 - S)(Bk_1 + Vk_2 + Sk_3) - \psi f(S)B \quad (41)$$

Remark: In the chemostat parameters scaling process we get un-dimensional parameters. If the system is a fixed point there is no nutrient, phage and bacteria change in time $\frac{dS}{dt} = 0; \frac{dV}{dt} = 0; \frac{dB}{dt} = 0$. We get three algebraic equations which give the expression for those three fixed point values. Our Chemostat system dynamic develops in time to bring us to fixed points. We sign the maximal rate bacteria reaction to nutrient as V_M, V_m , or V_{max} in our analysis. We define: $B(t) = B; S(t) = S; V(t) = V$

$$f_1(S, V, B) = (1 - S)(Bk_1 + Vk_2 + Sk_3) - \psi f(S)B \quad (42)$$

$$f_2(S, V, B) = -\varphi V + \gamma VB(1 - \frac{1}{\beta})$$

$$- V[Bk_1 + Vk_2 + Sk_3] \quad (43)$$

$$f_3(S, V, B) = \psi f(S)B - \gamma VB - B[Bk_1 + Vk_2 + Sk_3] \quad (44)$$

To find the fixed points for the system, we solve $\frac{dS}{dt} = 0; \frac{dV}{dt} = 0; \frac{dB}{dt} = 0$ i.e.:

$$(1 - S)(Bk_1 + Vk_2 + Sk_3) - \psi f(S)B = 0 \quad (45)$$

$$-\varphi V + \gamma VB(1 - \frac{1}{\beta}) - V[Bk_1 + Vk_2 + Sk_3] = 0 \quad (46)$$

$$\psi f(S)B - \gamma VB - B[Bk_1 + Vk_2 + Sk_3] = 0 \quad (47)$$

From the second equation $f_2(S, V, B) = 0$:

$$V\{-\varphi + \gamma B(1 - \frac{1}{\beta}) - [Bk_1 + Vk_2 + Sk_3]\} = 0 \quad (48)$$

$$V = 0 \text{ or } -\varphi + \gamma B(1 - \frac{1}{\beta}) - [Bk_1 + Vk_2 + Sk_3] = 0 \quad (49)$$

Case No.1: $V = 0$: Third equation gives:

$$\psi f(S)B - \gamma VB - B[Bk_1 + Vk_2 + Sk_3]$$

$$\psi f(S)B - B[Bk_1 + Sk_3] \quad (50)$$

$$\psi f(S)B - B[Bk_1 + Sk_3]$$

$$\Rightarrow B\{\psi f(S) - [Bk_1 + Sk_3]\} = 0 \quad (51)$$

Case No.1.1: Choosing $B = 0$ give from first equation

$$(1 - S)(Bk_1 + Vk_2 + Sk_3) - \psi f(S)B$$

$$(1 - S)(Sk_3) = 0 \Rightarrow S = 0 \text{ or } S = 1 \quad (52)$$

The first fixed point: $E^{(0)}(V^{(0)}, B^{(0)}, S^{(0)}) = (0, 0, 0)$

The second fixed point: $E^{(1)}(V^{(1)}, B^{(1)}, S^{(1)}) = (0, 0, 1)$

Case No.1.2: Choosing

$$\psi f(S) - [Bk_1 + Sk_3] = 0 \Rightarrow B = \frac{\psi f(S) - Sk_3}{k_1} \quad (53)$$

It gives from first equation

$$(1 - S)(Bk_1 + Vk_2 + Sk_3) - \psi f(S)B \Rightarrow V = 0 \quad (54)$$

$$(1 - S)([\frac{\psi f(S) - Sk_3}{k_1}]k_1 + Sk_3)$$

$$- \psi f(S)(\frac{\psi f(S) - Sk_3}{k_1}) = 0 \quad (55)$$

$$(1 - S)(\psi f(S) - Sk_3 + Sk_3)$$

$$- \psi f(S)(\frac{\psi f(S) - Sk_3}{k_1}) = 0 \quad (56)$$

$$(1 - S)[\psi f(S)] - \psi f(S)[\frac{\psi f(S) - Sk_3}{k_1}] = 0 \quad (57)$$

$$\psi f(S)\{1 - S - [\frac{\psi f(S) - Sk_3}{k_1}]\} = 0$$

$$f(S) = 0 \text{ or } (1 - S) - [\frac{\psi f(S) - Sk_3}{k_1}] = 0 \quad (58)$$

$$f(S) = 0 \Rightarrow \frac{V_{max}S}{a+S} \Rightarrow S = 0$$

$$B|_{S=0} = \frac{\psi f(S=0)}{k_1} = 0; f(S) = \frac{V_{max}S}{a+S} \quad (59)$$

$$(1 - S) - [\frac{\psi f(S) - Sk_3}{k_1}] = 0$$

$$1 - S - \frac{\psi}{k_1} f(S) + \frac{Sk_3}{k_1} = 0 \quad (60)$$

$$\begin{aligned} 1 - S - \frac{\psi}{k_1} f(S) + \frac{Sk_3}{k_1} &= 0 \\ 1 - S - \frac{\psi}{k_1} \left[\frac{V_{\max} S}{a+S} \right] + \frac{Sk_3}{k_1} &= 0 \end{aligned} \quad (61)$$

$$\begin{aligned} 1 + S \left[\frac{k_3}{k_1} - 1 \right] - \frac{\psi}{k_1} \left[\frac{V_{\max} S}{a+S} \right] &= 0 \\ a + S + S \left[\frac{k_3}{k_1} - 1 \right] (a+S) - \frac{\psi}{k_1} V_{\max} S &= 0 \end{aligned} \quad (62)$$

$$S^2 \left[\frac{k_3}{k_1} - 1 \right] + S \left\{ \left[\frac{k_3}{k_1} - 1 \right] a - \frac{\psi}{k_1} V_{\max} + 1 \right\} + a = 0 \quad (63)$$

$$\Phi_1(k_1, k_3, \psi, a) = \left\{ \left[\frac{k_3}{k_1} - 1 \right] a - \frac{\psi}{k_1} V_{\max} + 1 \right\}^2 - 4 \left[\frac{k_3}{k_1} - 1 \right] a \quad (64)$$

$$S^{(2,3)} = \frac{-\left\{ \left[\frac{k_3}{k_1} - 1 \right] a - \frac{\psi}{k_1} V_{\max} + 1 \right\} \pm \sqrt{\Phi_1(k_1, k_3, \psi, a)}}{2 \left[\frac{k_3}{k_1} - 1 \right]} \quad (65)$$

$$B^{(2,3)} = \frac{\psi f \left(\frac{\pm \sqrt{\Phi_1(k_1, k_3, \psi, a)}}{2 \left[\frac{k_3}{k_1} - 1 \right]} \right) - \left\{ \left[\frac{k_3}{k_1} - 1 \right] a - \frac{\psi}{k_1} V_{\max} + 1 \right\}}{2 \left[\frac{k_3}{k_1} - 1 \right]} k_3 \quad (66)$$

$$\left\{ \left[\frac{k_3}{k_1} - 1 \right] a - \frac{\psi}{k_1} V_{\max} + 1 \right\}^2 - 4 \left[\frac{k_3}{k_1} - 1 \right] a \Big|_{k_1=k_3} = \left(-\frac{\psi}{k_1} V_{\max} + 1 \right)^2 \quad (67)$$

$$\left\{ \left[\frac{k_3}{k_1} - 1 \right] a - \frac{\psi}{k_1} V_{\max} + 1 \right\} \Big|_{k_1=k_3} = 1 - \frac{\psi}{k_1} V_{\max} \quad (68)$$

The above case can't exist since $\frac{1}{2 \left[\frac{k_3}{k_1} - 1 \right]} \Big|_{k_1=k_3} \rightarrow \infty$.

The third and fourth fixed points:

$$\begin{aligned} E^{(2,3)}(V^{(2,3)}, B^{(2,3)}, S^{(2,3)}) \\ \psi f \left(\frac{\pm \sqrt{\Phi_1(k_1, k_3, \psi, a)}}{2 \left[\frac{k_3}{k_1} - 1 \right]} \right) \\ - \left\{ \left[\frac{k_3}{k_1} - 1 \right] a - \frac{\psi}{k_1} V_{\max} + 1 \right\} \\ - \left(\frac{\pm \sqrt{\Phi_1(k_1, k_3, \psi, a)}}{2 \left[\frac{k_3}{k_1} - 1 \right]} \right) k_3 \\ = (0, \frac{-\left\{ \left[\frac{k_3}{k_1} - 1 \right] a - \frac{\psi}{k_1} V_{\max} + 1 \right\} \pm \sqrt{\Phi_1(k_1, k_3, \psi, a)}}{2 \left[\frac{k_3}{k_1} - 1 \right]}) \end{aligned} \quad (69)$$

From the second equation $f_2(S, V, B)$:

$$V \left\{ -\varphi + \gamma B \left(1 - \frac{1}{\beta} \right) - [Bk_1 + Vk_2 + Sk_3] \right\} = 0 \quad (70)$$

Case No.2:

$$-\varphi + \gamma B \left(1 - \frac{1}{\beta} \right) - [Bk_1 + Vk_2 + Sk_3] = 0 \quad (71)$$

$$\begin{aligned} -\varphi + B \left[\gamma \left(1 - \frac{1}{\beta} \right) - k_1 \right] - Vk_2 - Sk_3 &= 0 \\ B = \frac{\varphi + Vk_2 + Sk_3}{\left[\gamma \left(1 - \frac{1}{\beta} \right) - k_1 \right]} \end{aligned} \quad (72)$$

Third equation gives:

$$\psi f(S)B - \gamma VB - B[Bk_1 + Vk_2 + Sk_3] = 0 \quad (73)$$

$$\begin{aligned} B \{ \psi f(S) - \gamma V - [Bk_1 + Vk_2 + Sk_3] \} &= 0 \\ \Rightarrow |_{B \neq 0} \psi f(S) - \gamma V - [Bk_1 + Vk_2 + Sk_3] &= 0 \end{aligned} \quad (74)$$

$$\psi f(S) - \gamma V - \left[\frac{\varphi + Vk_2 + Sk_3}{\left[\gamma \left(1 - \frac{1}{\beta} \right) - k_1 \right]} \right] k_1 + Vk_2 + Sk_3 = 0 \quad (75)$$

$$\psi f(S) - \gamma V - \left(\frac{\varphi + Vk_2 + Sk_3}{\left[\gamma \left(1 - \frac{1}{\beta} \right) - k_1 \right]} \right) k_1 - Vk_2 - Sk_3 = 0 \quad (76)$$

$$\begin{aligned} \psi f(S) - \gamma V - \frac{\varphi k_1}{\left[\gamma \left(1 - \frac{1}{\beta} \right) - k_1 \right]} - \frac{k_2 k_1}{\left[\gamma \left(1 - \frac{1}{\beta} \right) - k_1 \right]} V \\ - \frac{k_3 k_1}{\left[\gamma \left(1 - \frac{1}{\beta} \right) - k_1 \right]} S - Vk_2 - Sk_3 = 0 \end{aligned} \quad (77)$$

$$\begin{aligned} -\frac{k_2 k_1}{\left[\gamma \left(1 - \frac{1}{\beta} \right) - k_1 \right]} V - \gamma V - Vk_2 + \psi f(S) \\ - \frac{k_3 k_1}{\left[\gamma \left(1 - \frac{1}{\beta} \right) - k_1 \right]} S - Sk_3 - \frac{\varphi k_1}{\left[\gamma \left(1 - \frac{1}{\beta} \right) - k_1 \right]} = 0 \end{aligned} \quad (78)$$

$$\begin{aligned} -V \left[\frac{k_2 k_1}{\left[\gamma \left(1 - \frac{1}{\beta} \right) - k_1 \right]} + \gamma + k_2 \right] \\ + S \left\{ \frac{\psi V_{\max}}{a+S} - \frac{k_3 k_1}{\left[\gamma \left(1 - \frac{1}{\beta} \right) - k_1 \right]} - k_3 \right\} - \frac{\varphi k_1}{\left[\gamma \left(1 - \frac{1}{\beta} \right) - k_1 \right]} = 0 \end{aligned} \quad (79)$$

$$\begin{aligned} V \left[\frac{k_2 k_1}{\left[\gamma \left(1 - \frac{1}{\beta} \right) - k_1 \right]} + \gamma + k_2 \right] \\ = S \left\{ \frac{\psi V_{\max}}{a+S} - \left(\frac{k_3 k_1}{\left[\gamma \left(1 - \frac{1}{\beta} \right) - k_1 \right]} + k_3 \right) \right\} - \frac{\varphi k_1}{\left[\gamma \left(1 - \frac{1}{\beta} \right) - k_1 \right]} \end{aligned} \quad (80)$$

We define for simplicity:

$$\begin{aligned} \Gamma_1 &= \frac{k_2 k_1}{\left[\gamma \left(1 - \frac{1}{\beta} \right) - k_1 \right]} + \gamma + k_2 \\ \Gamma_2 &= \frac{k_3 k_1}{\left[\gamma \left(1 - \frac{1}{\beta} \right) - k_1 \right]} + k_3 \end{aligned} \quad (81)$$

$$\begin{aligned} \Gamma_3 &= \frac{\varphi k_1}{\left[\gamma \left(1 - \frac{1}{\beta} \right) - k_1 \right]}; V\Gamma_1 = S \left\{ \frac{\psi V_{\max}}{a+S} - \Gamma_2 \right\} - \Gamma_3 \\ V &= S \left\{ \frac{\psi V_{\max}}{a+S} - \Gamma_2 \right\} - \frac{\Gamma_3}{\Gamma_1} \end{aligned} \quad (82)$$

We have

$$B = \frac{\varphi + Vk_2 + Sk_3}{\left[\gamma \left(1 - \frac{1}{\beta} \right) - k_1 \right]}; V = S \left\{ \frac{\psi V_{\max}}{a+S} - \Gamma_2 \right\} - \frac{\Gamma_3}{\Gamma_1} \quad (83)$$

$$\begin{aligned} B &= \frac{\varphi + Vk_2 + Sk_3}{\left[\gamma \left(1 - \frac{1}{\beta} \right) - k_1 \right]} \\ B &= \frac{\varphi + S \left\{ \frac{\psi V_{\max}}{a+S} - \Gamma_2 \right\} - \frac{\Gamma_3}{\Gamma_1}}{\left[\gamma \left(1 - \frac{1}{\beta} \right) - k_1 \right]} k_2 + Sk_3 \end{aligned} \quad (84)$$

From first equation:

$$(1 - S)(Bk_1 + Vk_2 + Sk_3) - \psi f(S)B = 0 \quad (85)$$

$$(1-S)\left\{\frac{\varphi + [S \frac{\psi V_{\max} - \Gamma_2}{(a+S)} - \frac{\Gamma_3}{\Gamma_1}] k_2 + S k_3}{[\gamma(1-\frac{1}{\beta}) - k_1]}\right\} k_1 + \left\{S \frac{\psi V_{\max} - \Gamma_2}{\Gamma_1} - \frac{\Gamma_3}{\Gamma_1}\right\} k_2 + S k_3 - \psi f(S) \left\{\frac{\varphi + [S \frac{\psi V_{\max} - \Gamma_2}{(a+S)} - \frac{\Gamma_3}{\Gamma_1}] k_2 + S k_3}{[\gamma(1-\frac{1}{\beta}) - k_1]}\right\} = 0 \quad (86)$$

We define for simplicity $\Gamma_4 = [\gamma(1-\frac{1}{\beta}) - k_1]$

$$(1-S)\left\{\frac{\varphi + [S \frac{\psi V_{\max} - \Gamma_2}{(a+S)} - \frac{\Gamma_3}{\Gamma_1}] k_2 + S k_3}{\Gamma_4}\right\} k_1 + \left\{S \frac{\psi V_{\max} - \Gamma_2}{\Gamma_1} - \frac{\Gamma_3}{\Gamma_1}\right\} k_2 + S k_3 - \psi f(S) \left\{\frac{\varphi + [S \frac{\psi V_{\max} - \Gamma_2}{(a+S)} - \frac{\Gamma_3}{\Gamma_1}] k_2 + S k_3}{\Gamma_4}\right\} = 0 \quad (87)$$

$$(1-S)\left\{\frac{\varphi}{\Gamma_4} k_1 + [S \frac{\psi V_{\max} - \Gamma_2}{(a+S)} - \frac{\Gamma_3}{\Gamma_1}] \frac{1}{\Gamma_4} k_2 k_1 + S \frac{1}{\Gamma_4} k_3 k_1 + S \left(\frac{\psi V_{\max} - \Gamma_2}{(a+S)} - \frac{\Gamma_3}{\Gamma_1}\right) \frac{k_2}{\Gamma_1} - \frac{\Gamma_3}{\Gamma_1} k_2 + S k_3\right\} - \psi \frac{V_{\max} S}{a+S} \left\{\frac{\varphi}{\Gamma_4} + [S \frac{\psi V_{\max} - \Gamma_2}{(a+S)} - \frac{\Gamma_3}{\Gamma_1}] k_2 \frac{1}{\Gamma_4} + S k_3 \frac{1}{\Gamma_4}\right\} = 0; f(S) = \frac{V_{\max} S}{a+S} \quad (88)$$

For simplicity we define:

$$i(S) = \frac{\psi V_{\max} S}{(a+S)} \Rightarrow i(S) = \psi f(S) \Rightarrow \psi = \frac{i(S)}{f(S)} \quad (89)$$

$$(1-S)\left\{\frac{\varphi}{\Gamma_4} k_1 + \left[\frac{i(S) - S \Gamma_2}{\Gamma_1} - \frac{\Gamma_3}{\Gamma_1}\right] \frac{1}{\Gamma_4} k_2 k_1 + S \frac{1}{\Gamma_4} k_3 k_1 + [i(S) - S \Gamma_2] \frac{k_2}{\Gamma_1} - \frac{\Gamma_3}{\Gamma_1} k_2 + S k_3\right\} - i(S) \left\{\frac{\varphi}{\Gamma_4} + \left[\frac{i(S) - S \Gamma_2}{\Gamma_1} - \frac{\Gamma_3}{\Gamma_1}\right] k_2 \frac{1}{\Gamma_4} + S k_3 \frac{1}{\Gamma_4}\right\} = 0 \quad (90)$$

Since the above equation is polynomial shape in S variable, it is easy to solve it numerically than analytic to find fixed point $E^{(i>3)}(V^{(i>3)}, B^{(i>3)}, S^{(i>3)}); V^{(i>3)} \neq 0; B^{(i>3)} \neq 0; S^{(i>3)} \neq 0$. We need to neglect all fixed points numerical solutions which phage, bacteria or nutrient values are negative or complex [4] [5]. We can summary our last results in the next diagram:

$$\text{EQ.1} : (1-S)(B k_1 + V k_2 + S k_3) - \psi f(S) B = 0 \quad (91)$$

$$\text{EQ.2} : -\varphi V + \gamma V B (1 - \frac{1}{\beta}) - V[B k_1 + V k_2 + S k_3] = 0 \quad (92)$$

$$\text{EQ.3} : \psi f(S) B - \gamma V B - B[B k_1 + V k_2 + S k_3] = 0 \quad (93)$$

We already calculate our Chemostat scaled parameters and have the following values: $\gamma = 18.74; \psi = 3.69; a = 1.81; \varphi = \frac{\psi}{2} = 1.845; \beta = 71$. For our Chemostat washout rate balance parameters k_i [ml/sec].

$$V_M = 1; \sum_{i=1}^3 k_i = 1; \sum_{i=1}^3 k_i = 1 \quad (94)$$

$$k_1 = 0.2; k_2 = 0.3; k_3 = 0.5$$

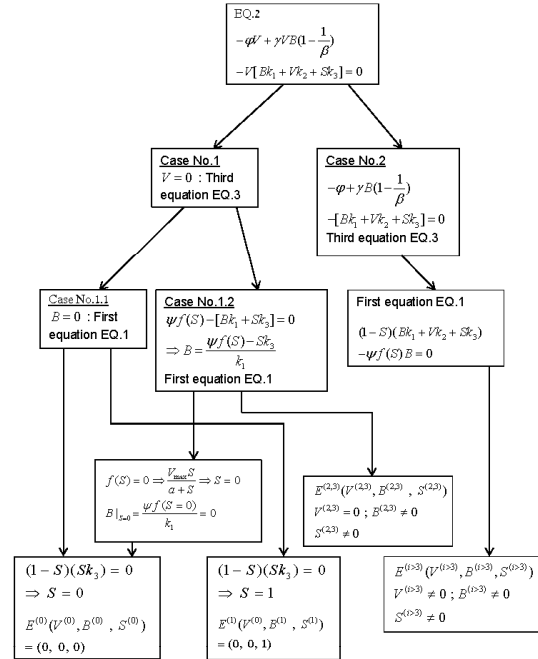


Fig. 2. Chemostat system with dilution control parameter fixed points flow chart.

$$\Gamma_1 = \frac{k_2 k_1}{[\gamma(1-\frac{1}{\beta}) - k_1]} + \gamma + k_2 \quad (95)$$

$$\Gamma_1 = \frac{0.3 \times 0.2}{[18.74 \times (1 - \frac{1}{71}) - 0.2]} + 18.74 + 0.3 = 19.04$$

$$\Gamma_2 = k_3 \left[\frac{k_1}{[\gamma(1-\frac{1}{\beta}) - k_1]} + 1 \right] \quad (96)$$

$$\Gamma_2 = 0.5 \times \left[\frac{0.2}{[18.74 \times (1 - \frac{1}{71}) - 0.2]} + 1 \right] = 0.505$$

$$\Gamma_3 = \frac{\varphi k_1}{[\gamma(1-\frac{1}{\beta}) - k_1]} = \frac{1.845 \times 0.2}{[18.74 \times (1 - \frac{1}{71}) - 0.2]} = 0.0202 \quad (97)$$

$$\Gamma_4 = [\gamma(1-\frac{1}{\beta}) - k_1] = 18.25$$

The first fixed point: $E^{(0)}(V^{(0)}, B^{(0)}, S^{(0)}) = (0, 0, 0)$

The second fixed point: $E^{(1)}(V^{(1)}, B^{(1)}, S^{(1)}) = (0, 0, 1)$

The third fixed point: $E^{(2)}(V^{(2)}, B^{(2)}, S^{(2)}) = (0, -8.63, 9.69)$

Only mathematic fixed point, not biologic (bacteria fixed point value is negative).

The fourth fixed point: $E^{(3)}(V^{(3)}, B^{(3)}, S^{(3)}) = (0, 0.885, 0.126)$

$$(1-S)\left\{\frac{\varphi}{\Gamma_4} k_1 + [S \frac{\psi V_{\max} - \Gamma_2}{(a+S)} - \frac{\Gamma_3}{\Gamma_1}] \frac{1}{\Gamma_4} k_2 k_1 + S \frac{1}{\Gamma_4} k_3 k_1 + S \left(\frac{\psi V_{\max} - \Gamma_2}{(a+S)} - \frac{\Gamma_3}{\Gamma_1}\right) \frac{k_2}{\Gamma_1} - \frac{\Gamma_3}{\Gamma_1} k_2 + S k_3\right\} - \psi \frac{V_{\max} S}{a+S} \left\{\frac{\varphi}{\Gamma_4} + [S \frac{\psi V_{\max} - \Gamma_2}{(a+S)} - \frac{\Gamma_3}{\Gamma_1}] k_2 \frac{1}{\Gamma_4} + S k_3 \frac{1}{\Gamma_4}\right\} = 0; f(S) = \frac{V_{\max} S}{a+S} \quad (98)$$

We use Mathematica software to solve the above equation (S variable):

$$\begin{cases} \{S = -1.79687 + 0.13257i\} \\ \{S = -1.79687 - 0.13257i\} \\ \{S = 0.676383\}, \{S = -0.0596224\} \end{cases} \quad (99)$$

We ignore mathematics results (Nutrient negative and complex values) and take only non-negative nutrient fixed point ($S \rightarrow 0.676383$).

$$V = S \frac{\{\frac{\psi V_{max}}{a+S} - \Gamma_2\}}{\Gamma_1} - \frac{\Gamma_3}{\Gamma_1} \quad (100)$$

$$V = 0.676 \times \frac{\{\frac{3.69}{(1.81+0.676)} - 0.505\}}{19.04}$$

$$- \frac{0.0202}{19.04} = 0.0337$$

$$B = \frac{\varphi + V k_2 + S k_3}{[\gamma(1 - \frac{1}{\beta}) - k_1]} \quad (101)$$

$$B = \frac{1.845 + 0.0337 \times 0.3 + 0.676 \times 0.5}{[18.74 \times (1 - \frac{1}{\beta}) - 0.2]} = 0.12$$

The fifth fixed point: $E^{(4)}(V^{(4)}, B^{(4)}, S^{(4)}) = (0.0337, 0.12, 0.676)$. In our final result we count only biological fixed points and neglect all mathematics fixed points. Our biological fixed points are as below.

The first fixed point: $E^{(0)}(V^{(0)}, B^{(0)}, S^{(0)}) = (0, 0, 0)$.

The second fixed point: $E^{(1)}(V^{(1)}, B^{(1)}, S^{(1)}) = (0, 0, 1)$.

The third fixed point: $E^{(2)}(V^{(2)}, B^{(2)}, S^{(2)}) = (0, 0.885, 0.126)$.

The fourth fixed point:

$E^{(3)}(V^{(3)}, B^{(3)}, S^{(3)}) = (0.0337, 0.12, 0.676)$.

II. CHEMOSTAT SYSTEM WITH DILUTION CONTROL PARAMETER FIXED POINTS CLASSIFICATION AND STABILITY

To classify Chemostat fixed points, we need to compute the Jacobian.

$$f_1(S, V, B) = (1 - S)(B k_1 + V k_2 + S k_3) - \psi f(S) B \quad (102)$$

$$f_2(S, V, B) = -\varphi V + \gamma V B (1 - \frac{1}{\beta}) - V [B k_1 + V k_2 + S k_3] \quad (103)$$

$$f_3(S, V, B) = \psi f(S) B - \gamma V B - B [B k_1 + V k_2 + S k_3] \quad (104)$$

We define $E^{(*)}(V^{(*)}, B^{(*)}, S^{(*)}) = (V^*, B^*, S^*)$ as any Chemostat fixed point.

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial S} & \frac{\partial f_1}{\partial V} & \frac{\partial f_1}{\partial B} \\ \frac{\partial f_2}{\partial S} & \frac{\partial f_2}{\partial V} & \frac{\partial f_2}{\partial B} \\ \frac{\partial f_3}{\partial S} & \frac{\partial f_3}{\partial V} & \frac{\partial f_3}{\partial B} \end{pmatrix} \quad ; \det |A - \lambda I| = 0$$

$$f(S) = \frac{V_{max} S}{a+S} \Rightarrow \frac{\partial f(S)}{\partial S} = \frac{V_{max} a}{(a+S)^2} \quad (105)$$

$$\frac{\partial f_1}{\partial S} |_{(V^*, B^*, S^*)} = -(B^* k_1 + V^* k_2 + S^* k_3) + (1 - S^*) k_3 - \psi B^* \frac{\partial f(S^*)}{\partial S} \quad (106)$$

$$\frac{\partial f_1}{\partial S} |_{(V^*, B^*, S^*)} = -(B^* k_1 + V^* k_2 + S^* k_3) + (1 - S^*) k_3 - \psi B^* \frac{V_{max} a}{(a+S^*)^2} \quad (107)$$

$$\begin{aligned} \frac{\partial f_1}{\partial V} |_{(V^*, B^*, S^*)} &= (1 - S^*) k_2 \\ \frac{\partial f_1}{\partial B} |_{(V^*, B^*, S^*)} &= (1 - S^*) k_1 - \psi f(S^*) \\ \frac{\partial f_2}{\partial S} |_{(V^*, B^*, S^*)} &= -V^* k_3 \end{aligned} \quad (108)$$

$$\begin{aligned} \frac{\partial f_2}{\partial V} |_{(V^*, B^*, S^*)} &= -\varphi + \gamma B^* (1 - \frac{1}{\beta}) \\ &- (B^* k_1 + V^* k_2 + S^* k_3) - V^* k_2 \end{aligned} \quad (109)$$

$$\begin{aligned} \frac{\partial f_2}{\partial B} |_{(V^*, B^*, S^*)} &= \gamma V^* (1 - \frac{1}{\beta}) - V^* k_1 \\ &= V^* [\gamma (1 - \frac{1}{\beta}) - k_1] \end{aligned} \quad (110)$$

$$\frac{\partial f_2}{\partial S} |_{(V^*, B^*, S^*)} = \psi B^* \frac{\partial f(S^*)}{\partial S} - B^* k_3$$

$$\begin{aligned} \frac{\partial f_3}{\partial S} |_{(V^*, B^*, S^*)} &= B^* [\psi \frac{V_{max} a}{(a+S^*)^2} - k_3] \\ \frac{\partial f_3}{\partial V} |_{(V^*, B^*, S^*)} &= -B^* (\gamma + k_2) \end{aligned} \quad (111)$$

$$\begin{aligned} \frac{\partial f_3}{\partial B} |_{(V^*, B^*, S^*)} &= \psi f(S^*) - \gamma V^* \\ &- (B^* k_1 + V^* k_2 + S^* k_3) - B^* k_1 \end{aligned} \quad (112)$$

$$\begin{aligned} \Xi_{11} &= \frac{\partial f_1}{\partial S} |_{(V^*, B^*, S^*)}; \Xi_{12} = \frac{\partial f_1}{\partial V} |_{(V^*, B^*, S^*)} \\ \Xi_{13} &= \frac{\partial f_1}{\partial B} |_{(V^*, B^*, S^*)}; \Xi_{21} = \frac{\partial f_2}{\partial S} |_{(V^*, B^*, S^*)} \\ \Xi_{22} &= \frac{\partial f_2}{\partial V} |_{(V^*, B^*, S^*)} \end{aligned} \quad (113)$$

$$\begin{aligned} \Xi_{23} &= \frac{\partial f_2}{\partial B} |_{(V^*, B^*, S^*)}; \Xi_{31} = \frac{\partial f_3}{\partial S} |_{(V^*, B^*, S^*)} \\ \Xi_{32} &= \frac{\partial f_3}{\partial V} |_{(V^*, B^*, S^*)}; \Xi_{33} = \frac{\partial f_3}{\partial B} |_{(V^*, B^*, S^*)} \end{aligned} \quad (114)$$

The Jacobian is:

$$A = \begin{pmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} \\ \Xi_{21} & \Xi_{22} & \Xi_{23} \\ \Xi_{31} & \Xi_{32} & \Xi_{33} \end{pmatrix}; \det |A - \lambda I| = 0$$

$$A - \lambda I = \begin{pmatrix} \Xi_{11} - \lambda & \Xi_{12} & \Xi_{13} \\ \Xi_{21} & \Xi_{22} - \lambda & \Xi_{23} \\ \Xi_{31} & \Xi_{32} & \Xi_{33} - \lambda \end{pmatrix} \quad (115)$$

$$\det |A - \lambda I| = 0 \Rightarrow (\Xi_{11} - \lambda)[(\Xi_{22} - \lambda)(\Xi_{33} - \lambda) - \Xi_{32}\Xi_{23}] - \Xi_{12}[\Xi_{21}(\Xi_{33} - \lambda) - \Xi_{31}\Xi_{23}] + \Xi_{13}[\Xi_{21}\Xi_{32} - \Xi_{31}(\Xi_{22} - \lambda)] \quad (116)$$

$$\begin{aligned} \Xi_{11} &= \frac{\partial f_1}{\partial S} |_{(V^*, B^*, S^*)}; \Xi_{12} = \frac{\partial f_1}{\partial V} |_{(V^*, B^*, S^*)} \\ \Xi_{13} &= \frac{\partial f_1}{\partial B} |_{(V^*, B^*, S^*)}; \Xi_{21} = \frac{\partial f_2}{\partial S} |_{(V^*, B^*, S^*)} \\ \Xi_{22} &= \frac{\partial f_2}{\partial V} |_{(V^*, B^*, S^*)} \end{aligned} \quad (117)$$

$$\begin{aligned} \Xi_{23} &= \frac{\partial f_2}{\partial B} |_{(V^*, B^*, S^*)}; \Xi_{31} = \frac{\partial f_3}{\partial S} |_{(V^*, B^*, S^*)} \\ \Xi_{32} &= \frac{\partial f_3}{\partial V} |_{(V^*, B^*, S^*)}; \Xi_{33} = \frac{\partial f_3}{\partial B} |_{(V^*, B^*, S^*)} \end{aligned} \quad (118)$$

$$A - \lambda I = \begin{pmatrix} -(B^*k_1 + V^*k_2 + S^*k_3) + (1-S^*)k_3 & (1-S^*)k_2 & (1-S^*)k_1 \\ -\psi B^* \frac{V_{max}^a}{(a+S^*)^2} - \lambda & -\varphi & -\psi f(S^*) \\ -V^*k_3 & +\gamma B^*(1-\frac{1}{\beta}) & V^*[\gamma(1-\frac{1}{\beta}) - k_1] \\ B^*\psi \frac{V_{max}^a}{(a+S^*)^2} - B^*k_3 & -V^*k_2 - \lambda & \psi f(S^*) - \gamma V^* \\ & & -(B^*k_1 + V^*k_2 + S^*k_3) \\ & & -B^*k_1 - \lambda \end{pmatrix} \quad (119)$$

We classify the first fixed point:
 $E^{(0)}(V^{(0)}, B^{(0)}, S^{(0)}) = (0, 0, 0)$.

$$\begin{aligned} \frac{\partial f_1}{\partial S} |_{(V^*, B^*, S^*)=(0,0,0)} &= k_3 \\ \frac{\partial f_1}{\partial V} |_{(V^*, B^*, S^*)=(0,0,0)} &= k_2 \\ \frac{\partial f_1}{\partial B} |_{(V^*, B^*, S^*)=(0,0,0)} &= k_1 \\ \frac{\partial f_1}{\partial S^2} |_{(V^*, B^*, S^*)=(0,0,0)} &= 0 \end{aligned} \quad (120)$$

$$\begin{aligned} \frac{\partial f_2}{\partial V} |_{(V^*, B^*, S^*)=(0,0,0)} &= -\varphi \\ \frac{\partial f_2}{\partial B} |_{(V^*, B^*, S^*)=(0,0,0)} &= 0 \\ \frac{\partial f_2}{\partial S} |_{(V^*, B^*, S^*)=(0,0,0)} &= 0 \end{aligned} \quad (121)$$

$$\begin{aligned} \frac{\partial f_3}{\partial S} |_{(V^*, B^*, S^*)=(0,0,0)} &= 0 \\ \frac{\partial f_3}{\partial V} |_{(V^*, B^*, S^*)=(0,0,0)} &= 0 \\ \frac{\partial f_3}{\partial B} |_{(V^*, B^*, S^*)=(0,0,0)} &= 0 \end{aligned} \quad (122)$$

$$\begin{aligned} \Xi_{11} = \frac{\partial f_1}{\partial S} |_{(V^*, B^*, S^*)} &; \Xi_{12} = \frac{\partial f_1}{\partial V} |_{(V^*, B^*, S^*)} \\ \Xi_{13} = \frac{\partial f_1}{\partial B} |_{(V^*, B^*, S^*)} &; \Xi_{21} = \frac{\partial f_2}{\partial S} |_{(V^*, B^*, S^*)} \\ \Xi_{22} = \frac{\partial f_2}{\partial V} |_{(V^*, B^*, S^*)} & \end{aligned} \quad (123)$$

$$\begin{aligned} \Xi_{23} = \frac{\partial f_2}{\partial B} |_{(V^*, B^*, S^*)} &; \Xi_{31} = \frac{\partial f_3}{\partial S} |_{(V^*, B^*, S^*)} \\ \Xi_{32} = \frac{\partial f_3}{\partial V} |_{(V^*, B^*, S^*)} &; \Xi_{33} = \frac{\partial f_3}{\partial B} |_{(V^*, B^*, S^*)} \end{aligned} \quad (124)$$

$$A - \lambda I |_{(V^*, B^*, S^*)=(0,0,0)} = \begin{pmatrix} k_3 - \lambda & k_2 & k_1 \\ 0 & -\varphi - \lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix}$$

$$\det |A - \lambda I |_{(V^*, B^*, S^*)=(0,0,0)} = 0 \quad (125)$$

$$\begin{aligned} \det |A - \lambda I |_{(V^*, B^*, S^*)=(0,0,0)} &= 0 \\ (k_3 - \lambda)(\varphi + \lambda)\lambda - k_2 \det \begin{pmatrix} 0 & 0 \\ 0 & -\lambda \end{pmatrix} & \\ + k_1 \det \begin{pmatrix} 0 & -\varphi - \lambda \\ 0 & 0 \end{pmatrix} &= 0 \end{aligned} \quad (126)$$

$$\begin{aligned} (k_3 - \lambda)(\varphi + \lambda)\lambda = 0 &\Rightarrow \lambda_1 = k_3 \\ \lambda_2 = -\varphi; \lambda_3 = 0; k_3 > 0; \varphi > 0 & \end{aligned} \quad (127)$$

Then the first fixed point is a saddle point.

We classify the second fixed point:
 $E^{(1)}(V^{(1)}, B^{(1)}, S^{(1)}) = (0, 0, 1)$

$$\begin{aligned} \frac{\partial f_1}{\partial S} |_{(V^*, B^*, S^*)=(0,0,1)} &= -k_3; f(S=1) = \frac{V_{max}}{a+1} \\ \frac{\partial f_1}{\partial V} |_{(V^*, B^*, S^*)=(0,0,1)} &= 0 \\ \frac{\partial f_1}{\partial B} |_{(V^*, B^*, S^*)=(0,0,1)} &= -\psi \frac{V_M}{a+1} \end{aligned} \quad (128)$$

$$\begin{aligned} \frac{\partial f_1}{\partial V} |_{(V^*, B^*, S^*)=(0,0,1)} &= 0 \\ \frac{\partial f_2}{\partial S} |_{(V^*, B^*, S^*)=(0,0,1)} &= 0 \\ \frac{\partial f_1}{\partial B} |_{(V^*, B^*, S^*)=(0,0,1)} &= -\psi f(S^* = 1) = -\psi \frac{V_M}{a+1} \end{aligned} \quad (129)$$

$$\begin{aligned} \frac{\partial f_2}{\partial V} |_{(V^*, B^*, S^*)=(0,0,1)} &= -\varphi - k_3 \\ \frac{\partial f_2}{\partial B} |_{(V^*, B^*, S^*)=(0,0,1)} &= 0 \\ \frac{\partial f_2}{\partial S} |_{(V^*, B^*, S^*)=(0,0,1)} &= 0 \end{aligned} \quad (130)$$

$$\begin{aligned} \frac{\partial f_3}{\partial S} |_{(V^*, B^*, S^*)=(0,0,1)} &= 0 \\ \frac{\partial f_3}{\partial V} |_{(V^*, B^*, S^*)=(0,0,1)} &= 0 \\ \frac{\partial f_3}{\partial B} |_{(V^*, B^*, S^*)=(0,0,1)} &= \psi \frac{V_{max}}{a+1} - k_3 \end{aligned} \quad (131)$$

$$\begin{aligned} \Xi_{11} = \frac{\partial f_1}{\partial S} |_{(V^*, B^*, S^*)} &; \Xi_{12} = \frac{\partial f_1}{\partial V} |_{(V^*, B^*, S^*)} \\ \Xi_{13} = \frac{\partial f_1}{\partial B} |_{(V^*, B^*, S^*)} &; \Xi_{21} = \frac{\partial f_2}{\partial S} |_{(V^*, B^*, S^*)} \\ \Xi_{22} = \frac{\partial f_2}{\partial V} |_{(V^*, B^*, S^*)} & \end{aligned} \quad (132)$$

$$\begin{aligned} \Xi_{23} = \frac{\partial f_2}{\partial B} |_{(V^*, B^*, S^*)} &; \Xi_{31} = \frac{\partial f_3}{\partial S} |_{(V^*, B^*, S^*)} \\ \Xi_{32} = \frac{\partial f_3}{\partial V} |_{(V^*, B^*, S^*)} &; \Xi_{33} = \frac{\partial f_3}{\partial B} |_{(V^*, B^*, S^*)} \end{aligned} \quad (133)$$

$$A - \lambda I = \begin{pmatrix} -k_3 - \lambda & 0 & -\psi \frac{V_M}{a+1} \\ 0 & -\varphi - k_3 - \lambda & 0 \\ 0 & 0 & \psi \frac{V_{max}}{a+1} - k_3 - \lambda \end{pmatrix} \quad (134)$$

$$\det |A - \lambda I| = (k_3 + \lambda)(\varphi + k_3 + \lambda)(\psi \frac{V_{max}}{a+1} - k_3 - \lambda) \quad (135)$$

We get three eigenvalues:

$$\lambda_1 = -k_3; \lambda_2 = -(\varphi + k_3); \lambda_3 = \psi \frac{V_{max}}{a+1} - k_3 \quad (136)$$

Fixed point status	Eigenvalues conditions	Parameters conditions
Stable node	$\lambda_1 < 0$ & $\lambda_2 < 0$ & $\lambda_3 < 0$	$k_3 > 0; \varphi + k_3 > 0$ $\psi \frac{V_{max}}{a+1} < k_3$ Can't exist for our parameters values.
Unstable node	$\lambda_1 > 0$ & $\lambda_2 > 0$ & $\lambda_3 > 0$	$k_3 < 0; \varphi + k_3 < 0$ $\psi \frac{V_{max}}{a+1} > k_3$ Can't exist for our parameters values.

Table 1a. Chemostat system with dilution control parameter fixed points and eigenvalues conditions.

We already calculate our chemostat scaled parameters in and have the following values:

$$\begin{aligned} \gamma = 18.74; \psi = 3.69; a = 1.81 \\ \varphi = \frac{\psi}{2} = 1.845; \beta = 71 \end{aligned} \quad (137)$$

For our Chemostat washout rate balance parameters

$$k_i[\text{ml/sec}] \text{ we choose } V_M = 1; \sum_{i=1}^3 k_i = 1$$

$$\sum_{i=1}^3 k_i = 1 \Rightarrow k_1 = 0.2; k_2 = 0.3; k_3 = 0.5 \quad (138)$$

$$(\psi_{\frac{V_{\max}}{a+1}} = 1.31) > (k_3 = 0.5).$$

Fixed point status	Eigenvalues conditions	Parameters conditions
Saddle point	$\lambda_1 < 0$ & $\lambda_2 < 0$ & $\lambda_3 > 0$	$k_3 > 0; \varphi + k_3 > 0$ $\psi_{\frac{V_{\max}}{a+1}} > k_3$ Can exist.
Saddle point	$\lambda_1 < 0$ & $\lambda_2 > 0$ & $\lambda_3 < 0$	$k_3 > 0; \varphi + k_3 < 0$ $\psi_{\frac{V_{\max}}{a+1}} < k_3$ Can't exist for our parameters values.
Saddle point	$\lambda_1 > 0$ & $\lambda_2 < 0$ & $\lambda_3 < 0$	$k_3 < 0; \varphi + k_3 > 0$ $\psi_{\frac{V_{\max}}{a+1}} < k_3$ Can't exist for our parameters values.

Table 1b. Chemostat system with dilution control parameter fixed points and eigenvalues conditions.

Fixed point status	Eigenvalues conditions	Parameters conditions
Saddle point	$\lambda_1 > 0$ & $\lambda_2 > 0$ & $\lambda_3 < 0$	$k_3 < 0; \varphi + k_3 < 0$ $\psi_{\frac{V_{\max}}{a+1}} < k_3$ Can't exist for our parameters values.
Saddle point	$\lambda_1 > 0$ & $\lambda_2 < 0$ & $\lambda_3 > 0$	$k_3 < 0; \varphi + k_3 > 0$ $\psi_{\frac{V_{\max}}{a+1}} > k_3$ Can't exist for our parameters values.
Saddle point	$\lambda_1 < 0$ & $\lambda_2 > 0$ & $\lambda_3 > 0$	$k_3 > 0; \varphi + k_3 < 0$ $\psi_{\frac{V_{\max}}{a+1}} > k_3$ Can't exist for our parameters values.

Table 1c. Chemostat system with dilution control parameter fixed points and eigenvalues conditions.

Second fixed point is a saddle point. We classify the third fixed point: $E^{(2)}(V^{(2)}, B^{(2)}, S^{(2)})=(0,0.885,0.126)$.

$$\frac{\partial f_1}{\partial S} |_{(V^*, B^*, S^*)=(0,0.885,0.126)} = -(0.885 \times k_1 + 0.126 \times k_3) + (1 - 0.126) \times k_3 \quad (139)$$

$$= -3.69 \times 0.885 \times \frac{1.81}{(1.81+0.126)^2}$$

$$\frac{\partial f_1}{\partial S} |_{(V^*, B^*, S^*)=(0,0.885,0.126)} = 0.748 \times k_3 - 0.885 \times k_1 - 1.58 \quad (140)$$

$$\frac{\partial f_1}{\partial V} |_{(V^*, B^*, S^*)=(0,0.885,0.126)} = 0.874 \times k_2$$

$$\frac{\partial f_1}{\partial B} |_{(V^*, B^*, S^*)=(0,0.885,0.126)} = 0.874 \times k_1 - 3.69 \times f(S^* = 0.126) = 0.874 \times k_1 - 0.24 \quad (141)$$

$$f(S^* = 0.126) = \frac{0.126}{1.81+0.126} = 0.065 \quad (142)$$

$$\frac{\partial f_2}{\partial S} |_{(V^*, B^*, S^*)=(0,0.885,0.126)} = 0$$

$$\frac{\partial f_2}{\partial V} |_{(V^*, B^*, S^*)=(0,0.885,0.126)} = -1.845 + 18.74 \times 0.885 \times (1 - \frac{1}{71}) - (0.885 \times k_1 + 0.126 \times k_3) \quad (143)$$

$$\frac{\partial f_2}{\partial B} |_{(V^*, B^*, S^*)=(0,0.885,0.126)} = 14.49 - 0.885 \times k_1 - 0.126 \times k_3 \quad (144)$$

$$\frac{\partial f_2}{\partial S} |_{(V^*, B^*, S^*)=(0,0.885,0.126)} = 0$$

$$\frac{\partial f_3}{\partial S} |_{(V^*, B^*, S^*)=(0,0.885,0.126)} = 0.885 \times [3.69 \times \frac{1.81}{(1.81+0.126)^2} - k_3] \quad (145)$$

$$= 0.885 \times [1.77 - k_3]$$

$$\frac{\partial f_3}{\partial S} |_{(V^*, B^*, S^*)=(0,0.885,0.126)} = B^* [\psi_{\frac{V_{\max}}{(a+S^*)^2}} - k_3] = 0.885 \times [1.77 - k_3] \quad (146)$$

$$\frac{\partial f_3}{\partial V} |_{(V^*, B^*, S^*)=(0,0.885,0.126)} = -0.885 \times (18.74 + k_2) \quad (147)$$

$$\frac{\partial f_3}{\partial B} |_{(V^*, B^*, S^*)=(0,0.885,0.126)} = \psi f(S^*) - 2B^* k_1 - S^* k_3 \quad (148)$$

$$= 0.24 - 1.77 \times k_1 - 0.126 \times k_3$$

A matrix element	A matrix element Function of k_i	A-Matrix element values for $\sum_{i=1}^3 k_i = 1$ $k_1 = 0.2$ $k_2 = 0.3$ $k_3 = 0.5$
$\frac{\partial f_1}{\partial S} _{(V^*, B^*, S^*)=(0,0.885,0.126)}$	$0.748 \times k_3 - 0.885 \times k_1 - 1.58$	-1.383
$\frac{\partial f_1}{\partial V} _{(V^*, B^*, S^*)=(0,0.885,0.126)}$	$0.874 \times k_2$	0.262
$\frac{\partial f_1}{\partial B} _{(V^*, B^*, S^*)=(0,0.885,0.126)}$	$0.874 \times k_1 - 0.24$	-0.0652
$\frac{\partial f_2}{\partial S} _{(V^*, B^*, S^*)=(0,0.885,0.126)}$	0	0
$\frac{\partial f_2}{\partial V} _{(V^*, B^*, S^*)=(0,0.885,0.126)}$	$14.49 - 0.885 \times k_1 - 0.126 \times k_3$	14.25
$\frac{\partial f_2}{\partial B} _{(V^*, B^*, S^*)=(0,0.885,0.126)}$	0	0
$\frac{\partial f_3}{\partial S} _{(V^*, B^*, S^*)=(0,0.885,0.126)}$	$0.885 \times [1.77 - k_3]$	1.12
$\frac{\partial f_3}{\partial V} _{(V^*, B^*, S^*)=(0,0.885,0.126)}$	$-0.885 \times (18.74 + k_2)$	-16.85
$\frac{\partial f_3}{\partial B} _{(V^*, B^*, S^*)=(0,0.885,0.126)}$	$0.24 - 1.77 \times k_1 - 0.126 \times k_3$	-0.177

Table 2. Chemostat system with dilution control parameter A matrix elements as a function of k_i .

$$A - \lambda I = \begin{pmatrix} \Xi_{11} - \lambda & \Xi_{12} & \Xi_{13} \\ \Xi_{21} & \Xi_{22} - \lambda & \Xi_{23} \\ \Xi_{31} & \Xi_{32} & \Xi_{33} - \lambda \end{pmatrix} \quad (149)$$

$$= \begin{pmatrix} -1.383 - \lambda & 0.262 & -0.0652 \\ 0 & 14.25 - \lambda & 0 \\ 1.12 & -16.85 & -0.177 - \lambda \end{pmatrix}$$

$$\det |A - \lambda I| = -(1.383 + \lambda)$$

$$\det \begin{pmatrix} 14.25 - \lambda & 0 \\ -16.85 & -(0.177 + \lambda) \end{pmatrix} \quad (150)$$

$$-0.0652 \det \begin{pmatrix} 0 & 14.25 - \lambda \\ 1.12 & -16.85 \end{pmatrix}$$

$$\det |A - \lambda I| = (1.383 + \lambda)(14.25 - \lambda)(0.177 + \lambda) + 0.0652 \times 1.12 \times (14.25 - \lambda) \quad (151)$$

$$\det |A - \lambda I| = (1.383 + \lambda)(14.25 - \lambda)(0.177 + \lambda) + 0.073(14.25 - \lambda) \Rightarrow \det |A - \lambda I| = 0 \quad (152)$$

$$\lambda_1 = -1.31802; \lambda_2 = -0.240978; \lambda_3 = 14.25 \quad (153)$$

Third fixed point is a saddle point.

$$\Xi_{11} = \frac{\partial f_1}{\partial S} |_{(V^*, B^*, S^*)}; \Xi_{12} = \frac{\partial f_1}{\partial V} |_{(V^*, B^*, S^*)}$$

$$\Xi_{13} = \frac{\partial f_1}{\partial B} |_{(V^*, B^*, S^*)}; \Xi_{21} = \frac{\partial f_2}{\partial S} |_{(V^*, B^*, S^*)} \quad (154)$$

$$\Xi_{22} = \frac{\partial f_2}{\partial V} |_{(V^*, B^*, S^*)}$$

$$\Xi_{23} = \frac{\partial f_2}{\partial B} |_{(V^*, B^*, S^*)}; \Xi_{31} = \frac{\partial f_3}{\partial S} |_{(V^*, B^*, S^*)} \quad (155)$$

$$\Xi_{32} = \frac{\partial f_3}{\partial V} |_{(V^*, B^*, S^*)}; \Xi_{33} = \frac{\partial f_3}{\partial B} |_{(V^*, B^*, S^*)}$$

We classify the fourth fixed point:

$$E^{(3)}(V^{(3)}, B^{(3)}, S^{(3)}) = (0.0337, 0.12, 0.676).$$

$$\frac{\partial f_1}{\partial S} |_{(V^*, B^*, S^*)} = -(B^*k_1 + V^*k_2 + S^*k_3)$$

$$= (0.0337, 0.12, 0.676)$$

$$+ (1 - S^*)k_3 - \psi B^* \frac{V_{\max} \alpha}{(\alpha + S^*)^2} \quad (156)$$

$$\frac{\partial f_1}{\partial V} |_{(V^*, B^*, S^*)} = -(0.12 \times 0.2 + 0.0337 \times 0.3$$

$$= (0.0337, 0.12, 0.676)$$

$$+ 0.676 \times 0.5) + (1 - S^*) \times 0.5$$

$$- 3.69 \times 0.12 \times \frac{1.81}{(1.81 + 0.676)^2} = -0.339 \quad (157)$$

$$\frac{\partial f_1}{\partial B} |_{(V^*, B^*, S^*)} = (1 - S^*)k_2 = 0.0972$$

$$\frac{\partial f_1}{\partial B} |_{(V^*, B^*, S^*)} = (1 - S^*)k_1 - \psi f(S^*)$$

$$= (1 - 0.676) \times 0.2 - 3.69 \times 0.271 = -0.925$$

$$f(S^* = 0.676) = \frac{0.676}{1.81 + 0.676} = 0.271 \quad (158)$$

$$\frac{\partial f_2}{\partial S} |_{(V^*, B^*, S^*)} = -V^*k_3 = -0.0337 \times 0.5 = -0.0168 \quad (159)$$

$$\frac{\partial f_2}{\partial V} |_{(V^*, B^*, S^*)} = -\varphi + \gamma B^* (1 - \frac{1}{\beta})$$

$$= (0.0337, 0.12, 0.676)$$

$$- (B^*k_1 + V^*k_2 + S^*k_3) - V^*k_2 \quad (160)$$

$$\frac{\partial f_2}{\partial V} |_{(V^*, B^*, S^*)} = -1.845 + 18.74 \times 0.12 \times (1 - \frac{1}{11})$$

$$= (0.0337, 0.12, 0.676)$$

$$- (0.12 \times 0.2 + 0.0337 \times 0.3 + 0.676 \times 0.5)$$

$$- 0.0337 \times 0.3 = -0.017 \quad (161)$$

$$\frac{\partial f_2}{\partial B} |_{(V^*, B^*, S^*)} = V^*[\gamma(1 - \frac{1}{\beta}) - k_1]$$

$$= 0.0337 \times [18.74 \times (1 - \frac{1}{11}) - 0.2] = 0.615$$

$$\frac{\partial f(S^*)}{\partial S} |_{S^*=0.676} = \frac{1.81}{(1.81 + 0.676)^2} = 0.292$$

$$\frac{\partial f_2}{\partial S} |_{(V^*, B^*, S^*)} = B^*[\psi \frac{\partial f(S^*)}{\partial S} - k_3]$$

$$= 0.12 \times [3.69 \times 0.292 - 0.5] = 0.0692 \quad (162)$$

$$\frac{\partial f_2}{\partial S} |_{(V^*, B^*, S^*)} = B^*[\psi \frac{V_{\max} \alpha}{(\alpha + S^*)^2} - k_3]$$

$$= (0.0337, 0.12, 0.676)$$

$$= 0.12 \times [3.69 \times \frac{1.81}{(1.81 + 0.676)^2} - 0.5] = 0.069$$

$$\frac{\partial f_2}{\partial V} |_{(V^*, B^*, S^*)} = -B^*(\gamma + k_2)$$

$$= (0.0337, 0.12, 0.676)$$

$$= -0.12 \times (18.74 + 0.3) = -2.248 \quad (163)$$

$$\frac{\partial f_2}{\partial B} |_{(V^*, B^*, S^*)} = \psi f(S^*) - \gamma V^*$$

$$= (0.0337, 0.12, 0.676)$$

$$- (B^*k_1 + V^*k_2 + S^*k_3) - B^*k_1 \quad (164)$$

$$\frac{\partial f_3}{\partial B} |_{(V^*, B^*, S^*)} = 3.69 \times 0.271 - 18.74 \times 0.0337$$

$$= (0.0337, 0.12, 0.676)$$

$$- (0.12 \times 0.2 + 0.0337 \times 0.3 + 0.676 \times 0.5)$$

$$- 0.12 \times 0.2 = -0.037$$

$$f(S^* = 0.676) = \frac{0.676}{1.81 + 0.676} = 0.271 \quad (165)$$

$$\Xi_{11} = \frac{\partial f_1}{\partial S} |_{(V^*, B^*, S^*)} = -0.339$$

$$\Xi_{12} = \frac{\partial f_1}{\partial V} |_{(V^*, B^*, S^*)} = 0.0972$$

$$\Xi_{13} = \frac{\partial f_1}{\partial B} |_{(V^*, B^*, S^*)} = -0.925 \quad (166)$$

$$\Xi_{21} = \frac{\partial f_2}{\partial S} |_{(V^*, B^*, S^*)} = -0.0168$$

$$\Xi_{22} = \frac{\partial f_2}{\partial V} |_{(V^*, B^*, S^*)} = -0.017$$

$$\Xi_{23} = \frac{\partial f_2}{\partial B} |_{(V^*, B^*, S^*)} = 0.615$$

$$\Xi_{31} = \frac{\partial f_3}{\partial S} |_{(V^*, B^*, S^*)} = 0.069$$

$$\Xi_{32} = \frac{\partial f_3}{\partial V} |_{(V^*, B^*, S^*)} = -2.248 \quad (167)$$

$$\Xi_{33} = \frac{\partial f_3}{\partial B} |_{(V^*, B^*, S^*)} = -0.037$$

$$A - \lambda I = \begin{pmatrix} -0.339 - \lambda & 0.0972 & -0.925 \\ -0.0168 & -0.017 - \lambda & 0.615 \\ 0.069 & -2.248 & -0.037 - \lambda \end{pmatrix}$$

$$\det |A - \lambda I| = 0 \quad (168)$$

$$\det|A - \lambda I| = -(0.339 + \lambda)$$

$$\det \begin{pmatrix} -(0.017 + \lambda) & 0.615 \\ -2.248 & -(0.037 + \lambda) \end{pmatrix}$$

$$-0.0972 \times \det \begin{pmatrix} -0.0168 & 0.615 \\ 0.069 & -(0.037 + \lambda) \end{pmatrix} \quad (169)$$

$$-0.925 \times \det \begin{pmatrix} -0.0168 & -(0.017 + \lambda) \\ 0.069 & -2.248 \end{pmatrix}$$

$$\lambda \rightarrow l$$

$$\det|A - \lambda I| = -(0.339 + \lambda)[(0.017 + \lambda)(0.037 + \lambda)$$

$$+ 2.248 \times 0.615] - 0.0972 \times [0.0168 \times (0.037 + \lambda)$$

$$- 0.069 \times 0.615] - 0.925 \times [0.0168 \times 2.248$$

$$+ 0.069 \times (0.017 + \lambda)] = 0; \lambda \rightarrow l \quad (170)$$

$$\det|A - \lambda I|$$

$$= -(0.339 + \lambda)[(0.017 + \lambda)(0.037 + \lambda) + 1.382]$$

$$- 0.0972 \times [0.0168 \times (0.037 + \lambda) - 0.0424]$$

$$- 0.925 \times [0.0377 + 0.069 \times (0.017 + \lambda)] = 0$$

$$\lambda \rightarrow l \quad (171)$$

Out: $\{l_1 = -0.345268\}, \{l_2 = -0.0238662 - 1.20389i\},$
 $\{l_3 = -0.0238662 + 1.20389i\}.$

Forth fixed point is a stable spiral. Remark: washout rate balance parameters (k_i) establish the classification of each Chemostat's fixed point. We need to plot our Chemostat system 3D diagram for different values of washout rate balance parameters (k_i) [6] [7].

MATLAB script:

$$V \rightarrow x(1); B \rightarrow x(2); S \rightarrow x(3)$$

$$X \rightarrow x(1); Y \rightarrow x(2); Z \rightarrow x(3) \quad (172)$$

Remark: There is a Basin Of Attraction (BOA) area surrounding forth fixed point (stable spiral). In the last two figures we choose initial values for bacteria, phage and nutrient within BOA area, all attracting values are positive and fit our forth fixed point coordinate values.

$$E^{(3)}(V^{(3)}, B^{(3)}, S^{(3)}) = (0.0337, 0.12, 0.676)$$

$$V \rightarrow X; B \rightarrow Y; S \rightarrow Z \quad (173)$$

III. CHEMOSTAT SYSTEM WITH CONTROL PARAMETER AND DELAYED VARIABLES ANALYSIS

Our system chemostat (1compartment gradostat) with control variables un-scaled equations: $\tau \neq 0; \tau > 0.$

$$\frac{dV(t)}{dt} = -\varphi V(t) - \gamma V(t)B(t)$$

$$+ \beta \gamma V(t - \tau)B(t - \tau) - V(t)D$$

$$\frac{dB(t)}{dt} = \psi f(S)B(t) - \gamma V(t)B(t) - B(t)D \quad (174)$$

$$\frac{dS(t)}{dt} = [S^{(0)} - S(t)]D - \psi f(S)B(t)/\delta$$

$$D = \xi(B, V, S) = Bk_1 + Vk_2 + Sk_3$$

$$V = V(t); B = B(t); S = S(t) \quad (175)$$

The scaled differential equations:

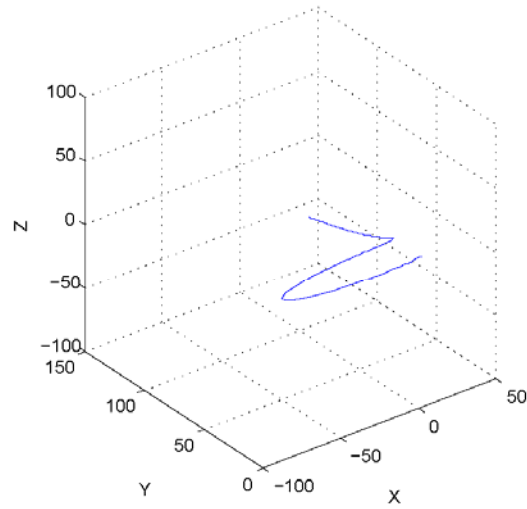


Fig. 3. Chemostat system with dilution control parameter $Z \vee Y \vee X$ for $k_1 = 0.2; k_2 = 0.3; k_3 = 0.5.$

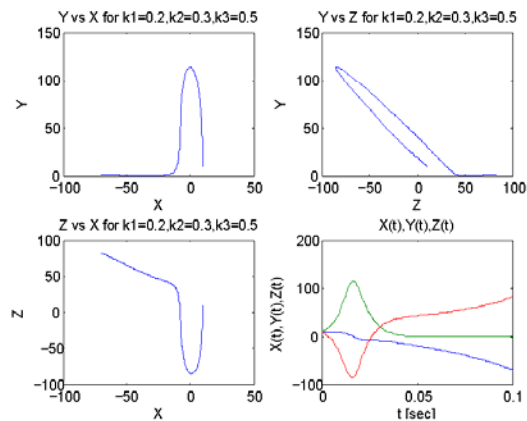


Fig. 4. Chemostat system with dilution control parameter variables phase spaces and time behavior for $k_1 = 0.2; k_2 = 0.3; k_3 = 0.5.$

$$\frac{dV}{dt} = -\varphi V - \gamma V B$$

$$+ \beta \gamma V(t - \tau)B(t - \tau) - V(t)D$$

$$\frac{dB}{dt} = \psi f(S)B - \gamma V B - B D \quad (176)$$

$$\frac{dS}{dt} = [S^{(0)} - S]D - \psi f(S)B/\delta$$

$$D = \xi(B, V, S) = Bk_1 + Vk_2 + Sk_3$$

Scaling of the above equation: All resources densities in term of $S^{(0)}$.

$$\frac{S}{S^{(0)}} \rightarrow S; \frac{V}{S^{(0)}\beta\delta} \rightarrow V; \frac{B}{S^{(0)}\delta} \rightarrow B$$

$$a \rightarrow \frac{a}{S^{(0)}}; t \rightarrow tD \quad (177)$$

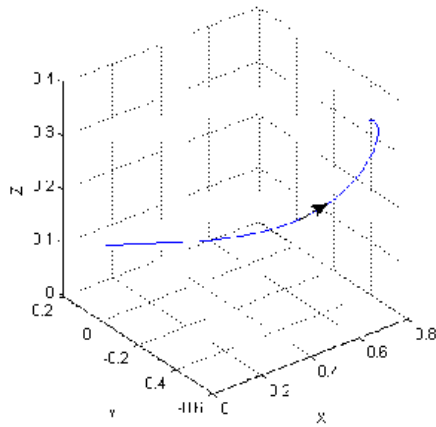


Fig. 5. Chemostat system with dilution control parameter $Z \vee Y \vee X$ for $k_1 = 100; k_2 = 80; k_3 = 70$.

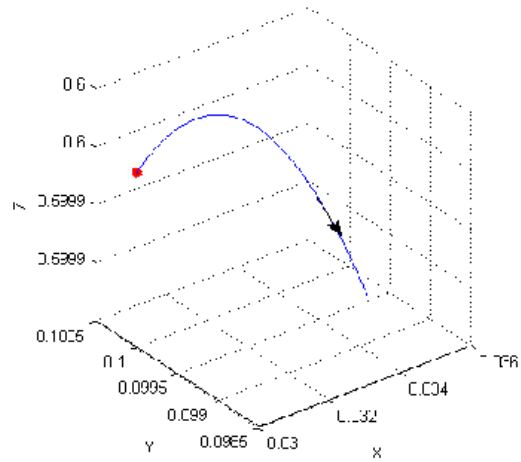


Fig. 7. Chemostat system with dilution control parameter $Z \vee Y \vee X$ for $k_1 = 0.2; k_2 = 0.8; k_3 = 0.5$.

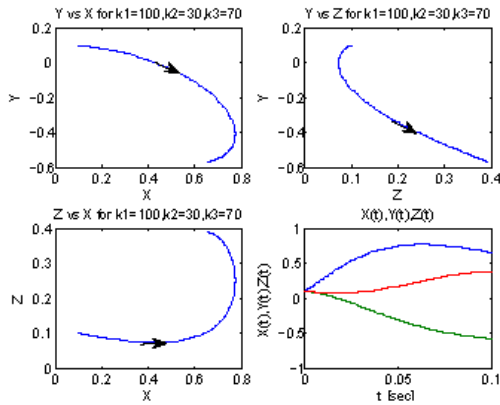


Fig. 6. Chemostat system with dilution control parameter variables phase spaces and time behavior for $k_1 = 100; k_2 = 30; k_3 = 70$.

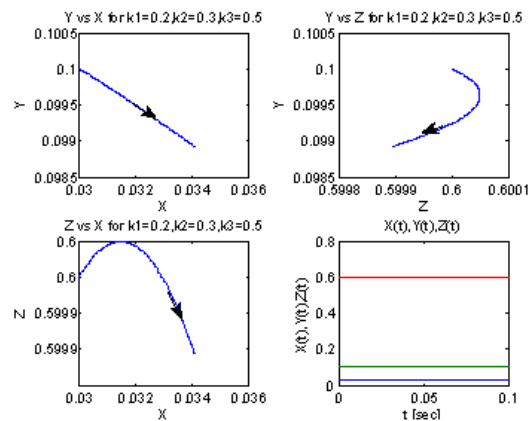


Fig. 8. Chemostat system with dilution control parameter variables phase spaces and time behavior for $k_1 = 0.2; k_2 = 0.3; k_3 = 0.5$.

Dimensional parameters have the following meaning:

$$\begin{aligned} \psi &\rightarrow \frac{\psi}{D}; \gamma \rightarrow \frac{\gamma S^{(0)} \delta}{D}; k_1 \rightarrow k_1 \delta S^{(0)} \\ k_2 &\rightarrow k_2 S^{(0)} \beta \delta; k_3 \rightarrow S^{(0)} k_3 \end{aligned} \quad (178)$$

$$\begin{aligned} S^{(0)} &\geq 0; B(0) > 0; \frac{S}{S^{(0)}} \rightarrow S \\ \frac{\delta}{S^{(0)}} &\rightarrow \alpha; f(S) = \frac{V_{max} S / S^{(0)}}{\alpha + S / S^{(0)}} \end{aligned} \quad (179)$$

$S^{(0)}$ term has units of concentration. $D[1/sec]$ has units of reciprocal time. We already scale the nutrient equation. The scales nutrient equation is:

$$\frac{dS}{dt} = (1 - S)(B k_1 + V k_2 + S k_3) - \psi f(S) B \quad (180)$$

The scaled bacteria equation:

$$\frac{dB}{dt} = \psi f(S) B - \gamma V B - B [B k_1 + V k_2 + S k_3] \quad (181)$$

Let take the phages differential equation:

$$\frac{V}{S^{(0)} \beta \delta} \rightarrow V; \frac{\varphi [\frac{1}{sec}]}{D [\frac{1}{sec}]} \rightarrow \varphi; \frac{B}{S^{(0)} \delta} \rightarrow B \quad (182)$$

$$\frac{dV}{dt} = -\varphi V - \gamma V B + \beta \gamma V (t - \tau) B (t - \tau) - V(t) D \quad (183)$$

Dimensional parameters have the following meaning:

$$\psi \rightarrow \frac{\psi}{D}; \gamma \rightarrow \frac{\gamma \beta S^{(0)} \delta}{D}; k_1 \rightarrow k_1 \delta S^{(0)} \quad (184)$$

$$k_2 \rightarrow k_2 S^{(0)} \beta \delta; k_3 \rightarrow S^{(0)} k_3$$

$$S^{(0)} \geq 0; B(0) > 0; \frac{S}{S^{(0)}} \rightarrow S; \frac{a}{S^{(0)}} \rightarrow a \quad (185)$$

$$f(S) = \frac{V_{max} S / S^{(0)}}{a / S^{(0)} + S / S^{(0)}}$$

$$D = Bk_1 + Vk_2 + Sk_3$$

$$\frac{dV}{dt} = -\varphi V - \gamma VB + \beta \gamma V(t - \tau) B(t - \tau) - V(t)[Bk_1 + Vk_2 + Sk_3] \quad (186)$$

$$\frac{d}{dt} \left[\frac{V}{S^{(0)} \beta \delta} \right] = -\varphi V \frac{1}{S^{(0)} \beta \delta} - \gamma V \frac{1}{S^{(0)} \beta \delta} B$$

$$+ \beta \gamma \frac{S^{(0)} \beta \delta}{D} V(t - \tau) \frac{1}{S^{(0)} \beta \delta} B(t - \tau) \frac{1}{S^{(0)} \delta} \frac{D}{\beta}$$

$$- V(t) \frac{1}{S^{(0)} \beta \delta} \left[\frac{B}{S^{(0)} \delta} k_1 S^{(0)} \delta + V \frac{1}{S^{(0)} \beta \delta} k_2 S^{(0)} \beta \delta + \frac{S}{S^{(0)}} k_3 S^{(0)} \right] \quad (187)$$

$$V(t - \tau) \frac{1}{S^{(0)} \beta \delta} \rightarrow V(t - \tau)$$

$$B(t - \tau) \frac{1}{S^{(0)} \delta} \rightarrow B(t - \tau) \quad (188)$$

$$\frac{d}{dt} \left[\frac{V}{S^{(0)} \beta \delta} \right] = -\varphi V \frac{1}{S^{(0)} \beta \delta} - \gamma \frac{S^{(0)} \beta \delta}{D} V \frac{1}{S^{(0)} \beta \delta} B \frac{1}{S^{(0)} \delta} \frac{D}{\beta}$$

$$+ \beta \gamma \frac{S^{(0)} \beta \delta}{D} V(t - \tau) \frac{1}{S^{(0)} \beta \delta} B(t - \tau) \frac{1}{S^{(0)} \delta} \frac{D}{\beta}$$

$$- V(t) \frac{1}{S^{(0)} \beta \delta} \left[\frac{B}{S^{(0)} \delta} k_1 S^{(0)} \delta + V \frac{1}{S^{(0)} \beta \delta} k_2 S^{(0)} \beta \delta + \frac{S}{S^{(0)}} k_3 S^{(0)} \right] \quad (189)$$

$$\frac{d}{dt} V = -\varphi V - \gamma VB \frac{D}{\beta} + \beta \gamma V(t - \tau) B(t - \tau) \frac{D}{\beta} - V(t)[Bk_1 + Vk_2 + Sk_3] \quad (190)$$

Measuring $S, a, B/\delta$ in units of $S^{(0)}$, and time in units of D^{-1}

$$\frac{d}{dt} V \frac{1}{D} = -\varphi \frac{1}{D} V - \gamma V B \frac{1}{\beta}$$

$$+ \gamma V(t - \tau) B(t - \tau)$$

$$- V(t)[Bk_1 + Vk_2 + Sk_3] \frac{1}{D} \quad (191)$$

$$\frac{d}{dt} V \frac{1}{D} \rightarrow \frac{d}{dt} V \quad (192)$$

$$[Bk_1 + Vk_2 + Sk_3] \frac{1}{D} \rightarrow [Bk_1 + Vk_2 + Sk_3] \quad (193)$$

$$\frac{B \left[\frac{1}{ml} \right]}{S^{(0)} \left[\frac{1}{ml} \right] \delta} k_1 \left[\frac{ml}{sec} \right] \delta S^{(0)} \left[\frac{1}{ml} \right] \rightarrow \left[\frac{1}{sec} \right]$$

$$\frac{V \left[\frac{1}{ml} \right]}{S^{(0)} \left[\frac{1}{ml} \right] \beta \delta} k_2 \left[\frac{ml}{sec} \right] S^{(0)} \left[\frac{1}{ml} \right] \beta \delta \rightarrow \left[\frac{1}{sec} \right] \quad (194)$$

$$\frac{S \left[\frac{1}{ml} \right]}{S^{(0)} \left[\frac{1}{ml} \right]} S^{(0)} \left[\frac{1}{ml} \right] k_3 \left[\frac{ml}{sec} \right] \rightarrow \left[\frac{1}{sec} \right]$$

$$\frac{\varphi \left[\frac{1}{sec} \right]}{D \left[\frac{1}{sec} \right]} \rightarrow \varphi \Rightarrow \frac{d}{dt} V = -\varphi V - \gamma V B \frac{1}{\beta} + \gamma V(t - \tau) B(t - \tau) - V(t)[Bk_1 + Vk_2 + Sk_3] \quad (195)$$

The scaled phage equation:

$$\frac{d}{dt} V = -\varphi V - \gamma V B \frac{1}{\beta} + \gamma V(t - \tau) B(t - \tau) - V(t)[Bk_1 + Vk_2 + Sk_3] \quad (196)$$

The Chemostat (One compartment Gradostat) with delay variables and control parameter scaled equations:

$$\frac{dS}{dt} = (1 - S)(Bk_1 + Vk_2 + Sk_3) - \psi f(S)B \quad (197)$$

$$\frac{d}{dt} V = -\varphi V - \gamma V B \frac{1}{\beta} + \gamma V(t - \tau) B(t - \tau) - V(t)[Bk_1 + Vk_2 + Sk_3] \quad (198)$$

$$\frac{dB}{dt} = \psi f(S)B - \gamma VB - B[Bk_1 + Vk_2 + Sk_3] \quad (199)$$

Stability analysis: The standard local stability analysis about any one of the equilibrium points of Chemostat system with control variables consists in adding to its coordinates $[VBS]$ arbitrarily small increments of exponential form $[vbs] e^{\lambda t}$, and retaining the first order terms in v, b, s . The system of three homogeneous equations leads to a polynomial characteristics equation in the eigenvalues λ . The polynomial characteristics equations accept by set the below phage, bacteria, and nutrient equations (delayed and un-delayed) into three Chemostat scaled equations. The phages, bacteria, and nutrient fixed values with arbitrarily small increments of exponential form $[vbs] e^{\lambda t}$ are: $i = 0$ (first fixed point), $i = 1$ (second fixed point), $i = 2$ (third fixed point) [2] [3].

$$V(t) = V^{(i)} + v e^{\lambda t}$$

$$B(t) = B^{(i)} + b e^{\lambda t}; S(t) = S^{(i)} + s e^{\lambda t}$$

$$V(t - \tau) = V^{(i)} + v e^{\lambda(t - \tau)}$$

$$B(t - \tau) = B^{(i)} + b e^{\lambda(t - \tau)}$$

$$\forall i = 0, 1, 2 \quad (200)$$

We choose the above expressions for our $V(t), B(t), S(t)$ as small displacement $[vbs]$ from the system fixed points at time $t = 0$.

$$V(t = 0) = V^{(i)} + v; B(t = 0) = B^{(i)} + b$$

$$S(t = 0) = S^{(i)} + s \quad (201)$$

for $\lambda < 0, t > 0$ the selected fixed point is stable otherwise $\lambda > 0, t > 0$ is Unstable. Our system tends to the selected fixed point exponentially for $\lambda < 0, t > 0$ otherwise go away from the selected fixed point exponentially. λ is the eigenvalue parameter which establish if the fixed point is stable or Unstable, additionally his absolute value ($|\lambda|$) establish the speed of flow toward or away from the selected fixed point [10] [11].

We already approve the equation for $f(S)$:

$$f(S) = \frac{[S^{(i)} + s e^{\lambda t}] a + [S^{(i)}]^2}{[a + S^{(i)}]^2} \quad (202)$$

First we take the Chemostat Nutrient differential equation:

$$\frac{dS}{dt} = (1 - S)(Bk_1 + Vk_2 + Sk_3) - \psi f(S)B \quad (203)$$

Adding to its coordinates $[VBS]$ arbitrarily small increments of exponential form $[vbs] e^{\lambda t}$, and retaining the first order terms in v, b, s .

$$\begin{aligned} S(t) &= S^{(i)} + s e^{\lambda t}; B(t) = B^{(i)} + b e^{\lambda t} \\ \frac{dS(t)}{dt} &= s \lambda e^{\lambda t}; V(t) = V^{(i)} + v e^{\lambda t} \\ \frac{dB(t)}{dt} &= v \lambda e^{\lambda t} \end{aligned} \quad (204)$$

$$\begin{aligned} s \lambda e^{\lambda t} &= (1 - [S^{(i)} + s e^{\lambda t}])([B^{(i)} + b e^{\lambda t}] k_1 \\ &+ [V^{(i)} + v e^{\lambda t}] k_2 + [S^{(i)} + s e^{\lambda t}] k_3) \\ &- \psi \left\{ \frac{[S^{(i)} + s e^{\lambda t}] a + [S^{(i)}]^2}{[a + S^{(i)}]^2} \right\} [B^{(i)} + b e^{\lambda t}] \end{aligned} \quad (205)$$

$$\begin{aligned} s \lambda e^{\lambda t} &= ([1 - S^{(i)}] - s e^{\lambda t})(B^{(i)} k_1 + b k_1 e^{\lambda t} \\ &+ V^{(i)} k_2 + v k_2 e^{\lambda t} + S^{(i)} k_3 + s k_3 e^{\lambda t}) \\ &- \psi \left\{ \frac{S^{(i)} a + s a e^{\lambda t} + [S^{(i)}]^2}{[a + S^{(i)}]^2} \right\} [B^{(i)} + b e^{\lambda t}] \end{aligned} \quad (206)$$

$$\begin{aligned} s \lambda e^{\lambda t} &= ([1 - S^{(i)}] - s e^{\lambda t})(B^{(i)} k_1 + b k_1 e^{\lambda t} \\ &+ V^{(i)} k_2 + v k_2 e^{\lambda t} + S^{(i)} k_3 + s k_3 e^{\lambda t}) \\ &- \left\{ \psi \frac{S^{(i)} a B^{(i)}}{[a + S^{(i)}]^2} + \psi \frac{S^{(i)} a b e^{\lambda t}}{[a + S^{(i)}]^2} + \psi \frac{s a e^{\lambda t} B^{(i)}}{[a + S^{(i)}]^2} \right. \\ &\left. + \psi \frac{s a e^{\lambda t} b e^{\lambda t}}{[a + S^{(i)}]^2} + \psi \frac{[S^{(i)}]^2 B^{(i)}}{[a + S^{(i)}]^2} + \psi \frac{[S^{(i)}]^2 b e^{\lambda t}}{[a + S^{(i)}]^2} \right\} \end{aligned} \quad (207)$$

$$\begin{aligned} s \lambda e^{\lambda t} &= ([1 - S^{(i)}] - s e^{\lambda t})(B^{(i)} k_1 + b k_1 e^{\lambda t} \\ &+ V^{(i)} k_2 + v k_2 e^{\lambda t} + S^{(i)} k_3 + s k_3 e^{\lambda t}) \\ &- \left\{ \psi \frac{S^{(i)} a B^{(i)}}{[a + S^{(i)}]^2} + \psi \frac{S^{(i)} a b e^{\lambda t}}{[a + S^{(i)}]^2} + \psi \frac{s a e^{\lambda t} B^{(i)}}{[a + S^{(i)}]^2} \right. \\ &\left. + \psi \frac{s a e^{\lambda t} b e^{\lambda t}}{[a + S^{(i)}]^2} + \psi \frac{[S^{(i)}]^2 B^{(i)}}{[a + S^{(i)}]^2} + \psi \frac{[S^{(i)}]^2 b e^{\lambda t}}{[a + S^{(i)}]^2} \right\} \end{aligned} \quad (208)$$

$$\begin{aligned} s \lambda e^{\lambda t} &= [1 - S^{(i)}][B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3] \\ &+ [1 - S^{(i)}][b k_1 e^{\lambda t} + v k_2 e^{\lambda t} + s k_3 e^{\lambda t}] \\ &- [B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3] s e^{\lambda t} \\ &- [b k_1 e^{\lambda t} + v k_2 e^{\lambda t} + s k_3 e^{\lambda t}] s e^{\lambda t} \\ &- \left\{ \psi \frac{S^{(i)} a B^{(i)}}{[a + S^{(i)}]^2} + \psi \frac{S^{(i)} a b e^{\lambda t}}{[a + S^{(i)}]^2} + \psi \frac{s a e^{\lambda t} B^{(i)}}{[a + S^{(i)}]^2} \right. \\ &\left. + \psi \frac{s a e^{\lambda t} b e^{\lambda t}}{[a + S^{(i)}]^2} + \psi \frac{[S^{(i)}]^2 B^{(i)}}{[a + S^{(i)}]^2} + \psi \frac{[S^{(i)}]^2 b e^{\lambda t}}{[a + S^{(i)}]^2} \right\} \end{aligned} \quad (209)$$

$sb \approx 0; vs \approx 0; s^2 \approx 0$ and the equation get the below expression :

$$\begin{aligned} s \lambda e^{\lambda t} &= [1 - S^{(i)}][B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3] \\ &+ [1 - S^{(i)}][b k_1 e^{\lambda t} + v k_2 e^{\lambda t} + s k_3 e^{\lambda t}] \\ &- [B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3] s e^{\lambda t} \\ &- \left\{ \psi \frac{S^{(i)} a B^{(i)}}{[a + S^{(i)}]^2} + \psi \frac{S^{(i)} a b e^{\lambda t}}{[a + S^{(i)}]^2} + \psi \frac{s a e^{\lambda t} B^{(i)}}{[a + S^{(i)}]^2} \right. \\ &\left. + \psi \frac{[S^{(i)}]^2 B^{(i)}}{[a + S^{(i)}]^2} + \psi \frac{[S^{(i)}]^2 b e^{\lambda t}}{[a + S^{(i)}]^2} \right\} \end{aligned} \quad (210)$$

Any Chemostat fixed point:

$$\frac{dS}{dt} = 0; f(S^{(i)}) = \frac{V_M S^{(i)}}{a + S^{(i)}}|_{V_M=1} = \frac{S^{(i)}}{a + S^{(i)}} \quad (211)$$

$$(1 - S^{(i)})(B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3) - \psi f(S^{(i)}) B^{(i)} = 0 \quad (212)$$

$$(1 - S^{(i)})(B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3) - \psi \frac{V_M S^{(i)}}{a + S^{(i)}} B^{(i)} = 0 \quad (213)$$

$$\begin{aligned} s \lambda e^{\lambda t} &= [1 - S^{(i)}][B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3] \\ &+ [1 - S^{(i)}][b k_1 e^{\lambda t} + v k_2 e^{\lambda t} + s k_3 e^{\lambda t}] \\ &- [B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3] s e^{\lambda t} \\ &- \left\{ \psi \frac{S^{(i)} a B^{(i)}}{[a + S^{(i)}]^2} + \psi \frac{S^{(i)} a b e^{\lambda t}}{[a + S^{(i)}]^2} + \psi \frac{s a e^{\lambda t} B^{(i)}}{[a + S^{(i)}]^2} \right. \\ &\left. + \psi \frac{s a e^{\lambda t} b e^{\lambda t}}{[a + S^{(i)}]^2} + \psi \frac{[S^{(i)}]^2 B^{(i)}}{[a + S^{(i)}]^2} \right\} \end{aligned} \quad (214)$$

$$\begin{aligned} &\psi \frac{S^{(i)} a B^{(i)}}{[a + S^{(i)}]^2} + \psi \frac{[S^{(i)}]^2 B^{(i)}}{[a + S^{(i)}]^2} \\ &= \psi \frac{B^{(i)} S^{(i)}}{[a + S^{(i)}]} = \psi B^{(i)} f(S^{(i)}) \end{aligned} \quad (215)$$

$$\begin{aligned} s \lambda e^{\lambda t} &= [1 - S^{(i)}][B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3] \\ &- \psi \frac{B^{(i)} S^{(i)}}{[a + S^{(i)}]} + [1 - S^{(i)}][b k_1 e^{\lambda t} + v k_2 e^{\lambda t} + s k_3 e^{\lambda t}] \\ &- [B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3] s e^{\lambda t} \\ &- \psi \frac{S^{(i)} a b e^{\lambda t}}{[a + S^{(i)}]^2} - \psi \frac{s a e^{\lambda t} B^{(i)}}{[a + S^{(i)}]^2} - \psi \frac{[S^{(i)}]^2 b e^{\lambda t}}{[a + S^{(i)}]^2} \end{aligned} \quad (216)$$

$$\begin{aligned} s \lambda e^{\lambda t} &= [1 - S^{(i)}][B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3] \\ &- \psi B^{(i)} f(S^{(i)}) + [1 - S^{(i)}][b k_1 e^{\lambda t} + v k_2 e^{\lambda t} + s k_3 e^{\lambda t}] \\ &- [B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3] s e^{\lambda t} - \psi \frac{S^{(i)} a b e^{\lambda t}}{[a + S^{(i)}]^2} \\ &- \psi \frac{s a e^{\lambda t} B^{(i)}}{[a + S^{(i)}]^2} - \psi \frac{[S^{(i)}]^2 b e^{\lambda t}}{[a + S^{(i)}]^2} \end{aligned} \quad (217)$$

$$(1 - S^{(i)})(B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3) - \psi f(S^{(i)}) B^{(i)} = 0 \quad (218)$$

Divide above equation by $e^{\lambda t}$ and we get

$$\begin{aligned} &[1 - S^{(i)}][b k_1 + v k_2 + s k_3] \\ &- [B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3] s \\ &- \psi \frac{S^{(i)} a b}{[a + S^{(i)}]^2} \\ &- \psi \frac{s a B^{(i)}}{[a + S^{(i)}]^2} - \psi \frac{[S^{(i)}]^2 b}{[a + S^{(i)}]^2} - s \lambda = 0 \end{aligned} \quad (219)$$

$$\begin{aligned} s \{ [1 - S^{(i)}] k_3 - [B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3] \\ - \psi \frac{a B^{(i)}}{[a + S^{(i)}]^2} - \lambda \} + v [1 - S^{(i)}] k_2 + b \{ [1 - S^{(i)}] k_1 \\ - \psi \frac{S^{(i)} a}{[a + S^{(i)}]^2} - \psi \frac{[S^{(i)}]^2}{[a + S^{(i)}]^2} \} = 0 \end{aligned} \quad (220)$$

We take the Chemostat Phage differential equation:

$$\begin{aligned} s \lambda e^{\lambda t} &= [1 - S^{(i)}][b k_1 e^{\lambda t} + v k_2 e^{\lambda t} + s k_3 e^{\lambda t}] \\ &- [B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3] s e^{\lambda t} \\ &- \psi \frac{S^{(i)} a b e^{\lambda t}}{[a + S^{(i)}]^2} - \psi \frac{s a e^{\lambda t} B^{(i)}}{[a + S^{(i)}]^2} - \psi \frac{[S^{(i)}]^2 b e^{\lambda t}}{[a + S^{(i)}]^2} \end{aligned} \quad (221)$$

$$\begin{aligned} [1 - S^{(i)}][b k_1 e^{\lambda t} + v k_2 e^{\lambda t} + s k_3 e^{\lambda t}] \\ - [B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3] s e^{\lambda t} - \psi \frac{S^{(i)} a b e^{\lambda t}}{[a + S^{(i)}]^2} \\ - \psi \frac{s a e^{\lambda t} B^{(i)}}{[a + S^{(i)}]^2} - \psi \frac{[S^{(i)}]^2 b e^{\lambda t}}{[a + S^{(i)}]^2} - s \lambda e^{\lambda t} = 0 \end{aligned} \quad (222)$$

$$\begin{aligned} \frac{d}{dt} V &= -\varphi V - \gamma V B \frac{1}{\beta} + \gamma V(t - \tau) B(t - \tau) \\ &- V(t) [B k_1 + V k_2 + S k_3] \end{aligned} \quad (223)$$

Adding to its coordinates $[VBS]$ arbitrarily small increments of exponential form $[vbs] e^{\lambda t}$, and retaining the first order terms in v, b, s .

$$\begin{aligned} S(t) &= S^{(i)} + s e^{\lambda t}; B(t) = B^{(i)} + b e^{\lambda t} \\ \frac{dS(t)}{dt} &= s \lambda e^{\lambda t}; V(t) = V^{(i)} + v e^{\lambda t} \\ \frac{dV(t)}{dt} &= v \lambda e^{\lambda t} \end{aligned} \quad (224) \quad v b \approx 0; s v \approx 0; v^2 \approx 0 \quad (234)$$

$$\begin{aligned} V(t - \tau) &= V^{(i)} + v e^{\lambda(t-\tau)} = V^{(i)} + v e^{\lambda t} e^{-\lambda \tau} \\ B(t - \tau) &= B^{(i)} + b e^{\lambda(t-\tau)} = B^{(i)} + b e^{\lambda t} e^{-\lambda \tau} \end{aligned} \quad (225)$$

$$\begin{aligned} v \lambda e^{\lambda t} &= -\varphi[V^{(i)} + v e^{\lambda t}] \\ &\quad -\gamma[V^{(i)} + v e^{\lambda t}][B^{(i)} + b e^{\lambda t}] \\ &\quad +\gamma[V^{(i)} + v e^{\lambda t} e^{-\lambda \tau}][B^{(i)} + b e^{\lambda t} e^{-\lambda \tau}] \\ &\quad -[V^{(i)} + v e^{\lambda t}]\{[B^{(i)} + b e^{\lambda t}]k_1 \\ &\quad +[V^{(i)} + v e^{\lambda t}]k_2 + [S^{(i)} + s e^{\lambda t}]k_3\} \end{aligned} \quad (226)$$

$$\begin{aligned} v \lambda e^{\lambda t} &= -\varphi V^{(i)} - \varphi v e^{\lambda t} - \gamma V^{(i)} B^{(i)} \frac{1}{\beta} \\ &\quad -\gamma V^{(i)} b e^{\lambda t} \frac{1}{\beta} - \gamma v e^{\lambda t} B^{(i)} \frac{1}{\beta} - \gamma v b e^{\lambda t} e^{\lambda t} \frac{1}{\beta} \\ &\quad +\gamma V^{(i)} B^{(i)} + \gamma V^{(i)} b e^{\lambda t} e^{-\lambda \tau} + \gamma v e^{\lambda t} e^{-\lambda \tau} B^{(i)} \\ &\quad +\gamma v b e^{\lambda t} e^{-\lambda \tau} e^{\lambda t} e^{-\lambda \tau} - [V^{(i)} + v e^{\lambda t}]\{B^{(i)} k_1 \\ &\quad + b k_1 e^{\lambda t} + V^{(i)} k_2 + v k_2 e^{\lambda t} + S^{(i)} k_3 + s k_3 e^{\lambda t}\} \end{aligned} \quad (227)$$

$$\begin{aligned} v \lambda e^{\lambda t} &= -\varphi V^{(i)} - \varphi v e^{\lambda t} - \gamma V^{(i)} B^{(i)} \frac{1}{\beta} \\ &\quad -\gamma V^{(i)} b e^{\lambda t} \frac{1}{\beta} - \gamma v e^{\lambda t} B^{(i)} \frac{1}{\beta} - \gamma v b e^{\lambda t} e^{\lambda t} \frac{1}{\beta} \\ &\quad +\gamma V^{(i)} B^{(i)} + \gamma V^{(i)} b e^{\lambda t} e^{-\lambda \tau} + \gamma v e^{\lambda t} e^{-\lambda \tau} B^{(i)} \\ &\quad +\gamma v b e^{\lambda t} e^{-\lambda \tau} e^{\lambda t} e^{-\lambda \tau} - [V^{(i)} + v e^{\lambda t}]\{B^{(i)} k_1 \\ &\quad + b k_1 e^{\lambda t} + V^{(i)} k_2 + v k_2 e^{\lambda t} + S^{(i)} k_3 + s k_3 e^{\lambda t}\} \end{aligned} \quad (228)$$

$$\begin{aligned} v \lambda e^{\lambda t} &= -\varphi V^{(i)} - \varphi v e^{\lambda t} - \gamma V^{(i)} B^{(i)} \frac{1}{\beta} \\ &\quad -\gamma V^{(i)} b e^{\lambda t} \frac{1}{\beta} - \gamma v e^{\lambda t} B^{(i)} \frac{1}{\beta} - \gamma v b e^{\lambda t} e^{\lambda t} \frac{1}{\beta} \\ &\quad +\gamma V^{(i)} B^{(i)} + \gamma V^{(i)} b e^{\lambda t} e^{-\lambda \tau} + \gamma v e^{\lambda t} e^{-\lambda \tau} B^{(i)} \\ &\quad +\gamma v b e^{\lambda t} e^{-\lambda \tau} e^{\lambda t} e^{-\lambda \tau} - V^{(i)}[B^{(i)} k_1 + V^{(i)} k_2 \\ &\quad + S^{(i)} k_3] - V^{(i)}[b k_1 + v k_2 + s k_3] e^{\lambda t} - [B^{(i)} k_1 \\ &\quad + V^{(i)} k_2 + S^{(i)} k_3] v e^{\lambda t} - [b k_1 + v k_2 \\ &\quad + s k_3] e^{\lambda t} v e^{\lambda t} \end{aligned} \quad (229)$$

$$\begin{aligned} v \lambda e^{\lambda t} &= -\varphi V^{(i)} - \gamma V^{(i)} B^{(i)} \frac{1}{\beta} + \gamma V^{(i)} B^{(i)} \\ &\quad - V^{(i)}[B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3] - \varphi v e^{\lambda t} \\ &\quad -\gamma V^{(i)} b e^{\lambda t} \frac{1}{\beta} - \gamma v e^{\lambda t} B^{(i)} \frac{1}{\beta} - \gamma v b e^{\lambda t} e^{\lambda t} \frac{1}{\beta} \\ &\quad +\gamma V^{(i)} b e^{\lambda t} e^{-\lambda \tau} + \gamma v e^{\lambda t} e^{-\lambda \tau} B^{(i)} \\ &\quad +\gamma v b e^{\lambda t} e^{-\lambda \tau} e^{\lambda t} e^{-\lambda \tau} - V^{(i)}[b k_1 + v k_2 + s k_3] e^{\lambda t} \\ &\quad - [B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3] v e^{\lambda t} \\ &\quad - [b k_1 + v k_2 + s k_3] e^{\lambda t} v e^{\lambda t} \end{aligned} \quad (230)$$

Any Chemostat fixed point:

$$\lim_{t \rightarrow \infty} V(t - \tau) = \lim_{t \rightarrow \infty} V(t); \lim_{t \rightarrow \infty} B(t - \tau) = \lim_{t \rightarrow \infty} B(t) \quad (231)$$

$$\begin{aligned} \frac{d}{dt} V = 0 &\Rightarrow -\varphi V^{(i)} - \gamma V^{(i)} B^{(i)} \frac{1}{\beta} + \gamma V^{(i)} B^{(i)} \\ -V^{(i)}[B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3] &= 0 \end{aligned} \quad (232)$$

$$\begin{aligned} v \lambda e^{\lambda t} &= -\varphi v e^{\lambda t} - \gamma V^{(i)} b e^{\lambda t} \frac{1}{\beta} - \gamma v e^{\lambda t} B^{(i)} \frac{1}{\beta} \\ &\quad -\gamma v b e^{\lambda t} e^{\lambda t} \frac{1}{\beta} + \gamma V^{(i)} b e^{\lambda t} e^{-\lambda \tau} + \gamma v e^{\lambda t} e^{-\lambda \tau} B^{(i)} \\ &\quad +\gamma v b e^{\lambda t} e^{-\lambda \tau} e^{\lambda t} e^{-\lambda \tau} - V^{(i)}[b k_1 + v k_2 + s k_3] e^{\lambda t} \\ &\quad - [B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3] v e^{\lambda t} \\ &\quad - [b v k_1 + v v k_2 + s v k_3] e^{2 \lambda t} \end{aligned} \quad (233)$$

$$\begin{aligned} v \lambda e^{\lambda t} &= -\varphi v e^{\lambda t} - \gamma V^{(i)} b e^{\lambda t} \frac{1}{\beta} - \gamma v e^{\lambda t} B^{(i)} \frac{1}{\beta} \\ &\quad +\gamma V^{(i)} b e^{\lambda t} e^{-\lambda \tau} + \gamma v e^{\lambda t} e^{-\lambda \tau} B^{(i)} \\ &\quad - V^{(i)}[b k_1 + v k_2 + s k_3] e^{\lambda t} - [B^{(i)} k_1 + V^{(i)} k_2 \\ &\quad + S^{(i)} k_3] v e^{\lambda t} \end{aligned} \quad (235)$$

$$\begin{aligned} -\varphi v e^{\lambda t} - \gamma V^{(i)} b e^{\lambda t} \frac{1}{\beta} - \gamma v e^{\lambda t} B^{(i)} \frac{1}{\beta} \\ +\gamma V^{(i)} b e^{\lambda t} e^{-\lambda \tau} + \gamma v e^{\lambda t} e^{-\lambda \tau} B^{(i)} \\ - V^{(i)}[b k_1 + v k_2 + s k_3] e^{\lambda t} - [B^{(i)} k_1 + V^{(i)} k_2 + \\ S^{(i)} k_3] v e^{\lambda t} = 0 \end{aligned} \quad (236)$$

$$\begin{aligned} -s V^{(i)} k_3 e^{\lambda t} + v\{\gamma e^{\lambda t} e^{-\lambda \tau} B^{(i)} - \varphi e^{\lambda t} \\ -\gamma e^{\lambda t} B^{(i)} \frac{1}{\beta} - \lambda e^{\lambda t} - V^{(i)} k_2 e^{\lambda t} \\ - [B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3] e^{\lambda t}\} \\ + b\{\gamma V^{(i)} e^{\lambda t} e^{-\lambda \tau} - \gamma V^{(i)} e^{\lambda t} \frac{1}{\beta} \\ - V^{(i)} k_1 e^{\lambda t}\} = 0 \end{aligned} \quad (237)$$

Divide above equation by $e^{\lambda t}$ and we get

$$\begin{aligned} -s V^{(i)} k_3 + v\{\gamma e^{-\lambda \tau} B^{(i)} - \varphi - \gamma B^{(i)} \frac{1}{\beta} \\ - V^{(i)} k_2 - [B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3] - \lambda\} \\ + b\{\gamma V^{(i)} e^{-\lambda \tau} - \gamma V^{(i)} \frac{1}{\beta} - V^{(i)} k_1\} = 0 \end{aligned} \quad (238)$$

We take the Chemostat bacteria differential equation:

$$\frac{dB}{dt} = \psi f(S)B - \gamma VB - B[Bk_1 + V k_2 + S k_3] \quad (239)$$

Adding to its coordinates $[VBS]$ arbitrarily small increments of exponential form $[vbs] e^{\lambda t}$, and retaining the first order terms in v, b, s .

$$\begin{aligned} B(t) &= B^{(i)} + b e^{\lambda t}; \frac{dB(t)}{dt} = b \lambda e^{\lambda t} \\ \frac{dS(t)}{dt} &= s \lambda e^{\lambda t}; V(t) = V^{(i)} + v e^{\lambda t} \\ \frac{dV(t)}{dt} &= v \lambda e^{\lambda t}; S(t) = S^{(i)} + s e^{\lambda t} \end{aligned} \quad (240)$$

$$\begin{aligned} V(t - \tau) &= V^{(i)} + v e^{\lambda(t-\tau)} = V^{(i)} + v e^{\lambda t} e^{-\lambda \tau} \\ B(t - \tau) &= B^{(i)} + b e^{\lambda(t-\tau)} = B^{(i)} + b e^{\lambda t} e^{-\lambda \tau} \end{aligned} \quad (241)$$

$$\begin{aligned} f(S) &= \frac{[S^{(i)} + s e^{\lambda t}]a + [S^{(i)}]^2}{[a + S^{(i)}]^2} \\ f(S^{(i)})|_{V_M=1} &= \frac{S^{(i)}}{(a + S^{(i)})} \end{aligned} \quad (242)$$

$$\begin{aligned} b \lambda e^{\lambda t} &= \psi \left[\frac{[S^{(i)} + s e^{\lambda t}]a + [S^{(i)}]^2}{[a + S^{(i)}]^2} \right] [B^{(i)} + b e^{\lambda t}] \\ &\quad -\gamma [V^{(i)} + v e^{\lambda t}][B^{(i)} + b e^{\lambda t}] - [B^{(i)} + b e^{\lambda t}][B^{(i)} \\ &\quad + b e^{\lambda t}]k_1 + (V^{(i)} + v e^{\lambda t})k_2 + (S^{(i)} + s e^{\lambda t})k_3 \end{aligned} \quad (243)$$

$$\begin{aligned} b \lambda e^{\lambda t} &= \psi \left[\frac{S^{(i)}}{(a + S^{(i)})} + \frac{s a e^{\lambda t}}{[a + S^{(i)}]^2} \right] [B^{(i)} + b e^{\lambda t}] \\ &\quad -\gamma [V^{(i)} B^{(i)} + V^{(i)} b e^{\lambda t} + B^{(i)} v e^{\lambda t} \\ &\quad + v b e^{2 \lambda t}] - [B^{(i)} + b e^{\lambda t}][B^{(i)} k_1 + V^{(i)} k_2 \\ &\quad + S^{(i)} k_3] + (b k_1 + v k_2 + s k_3) e^{\lambda t} \end{aligned} \quad (244)$$

$$\begin{aligned}
 b\lambda e^{\lambda t} &= \psi f(S^{(i)}) B^{(i)} + \psi f(S^{(i)}) b e^{\lambda t} \\
 &+ \psi B^{(i)} \frac{sa e^{\lambda t}}{[a+S^{(i)}]^2} + \psi \frac{sa e^{\lambda t} b e^{\lambda t}}{[a+S^{(i)}]^2} \\
 &- \gamma [V^{(i)} B^{(i)} + V^{(i)} b e^{\lambda t} + B^{(i)} v e^{\lambda t} + vb e^{2\lambda t}] \\
 &- B^{(i)} (B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3) \\
 &- B^{(i)} (bk_1 + vk_2 + sk_3) e^{\lambda t} - (B^{(i)} k_1 + V^{(i)} k_2 \\
 &+ S^{(i)} k_3) b e^{\lambda t} - (bk_1 + vk_2 + sk_3) e^{\lambda t} b e^{\lambda t}
 \end{aligned} \tag{245}$$

$$vs \approx 0; vb \approx 0; sb \approx 0; sv \approx 0; b^2 \approx 0 \tag{246}$$

$$\begin{aligned}
 b\lambda e^{\lambda t} &= \psi f(S^{(i)}) B^{(i)} + \psi f(S^{(i)}) b e^{\lambda t} \\
 &+ \psi_1 B^{(i)} \frac{sa e^{\lambda t}}{[a+S^{(i)}]^2} - \gamma [V^{(i)} B^{(i)} + V^{(i)} b e^{\lambda t} \\
 &+ B^{(i)} v e^{\lambda t}] - B^{(i)} (B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3) \\
 &- B^{(i)} (bk_1 + vk_2 + sk_3) e^{\lambda t} - (B^{(i)} k_1 + V^{(i)} k_2 \\
 &+ S^{(i)} k_3) b e^{\lambda t}
 \end{aligned} \tag{247}$$

$$\begin{aligned}
 b\lambda e^{\lambda t} &= \psi f(S^{(i)}) B^{(i)} - \gamma V^{(i)} B^{(i)} - B^{(i)} (B^{(i)} k_1 \\
 &+ V^{(i)} k_2 + S^{(i)} k_3) + \psi f(S^{(i)}) b e^{\lambda t} + \psi B^{(i)} \frac{sa e^{\lambda t}}{[a+S^{(i)}]^2} \\
 &- \gamma V^{(i)} b e^{\lambda t} - \gamma B^{(i)} v e^{\lambda t} - B^{(i)} (bk_1 + vk_2 + sk_3) e^{\lambda t} \\
 &- (B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3) b e^{\lambda t}
 \end{aligned} \tag{248}$$

Any Chemostat fixed point: $\frac{dB}{dt} = 0$:

$$\begin{aligned}
 \frac{dB}{dt} = 0 &\Rightarrow \psi f(S^{(i)}) B^{(i)} - \gamma V^{(i)} B^{(i)} \\
 &- B^{(i)} [B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3] = 0
 \end{aligned} \tag{249}$$

$$\begin{aligned}
 b\lambda e^{\lambda t} &= \psi f(S^{(i)}) b e^{\lambda t} + \psi B^{(i)} \frac{sa e^{\lambda t}}{[a+S^{(i)}]^2} \\
 &- \gamma V^{(i)} b e^{\lambda t} - \gamma B^{(i)} v e^{\lambda t} - B^{(i)} (bk_1 \\
 &+ vk_2 + sk_3) e^{\lambda t} - (B^{(i)} k_1 + V^{(i)} k_2 \\
 &+ S^{(i)} k_3) b e^{\lambda t}
 \end{aligned} \tag{250}$$

$$\begin{aligned}
 \psi f(S^{(i)}) b e^{\lambda t} &+ \psi B^{(i)} \frac{sa e^{\lambda t}}{[a+S^{(i)}]^2} - \gamma V^{(i)} b e^{\lambda t} \\
 &- \gamma B^{(i)} v e^{\lambda t} - B^{(i)} (bk_1 + vk_2 + sk_3) e^{\lambda t} \\
 &- (B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3) b e^{\lambda t} - b\lambda e^{\lambda t} = 0
 \end{aligned} \tag{251}$$

$$\begin{aligned}
 s(\psi B^{(i)} \frac{a e^{\lambda t}}{[a+S^{(i)}]^2} - B^{(i)} k_3 e^{\lambda t}) &- v(B^{(i)} k_2 e^{\lambda t} \\
 &+ \gamma B^{(i)} e^{\lambda t}) + b\{\psi f(S^{(i)}) e^{\lambda t} - \gamma V^{(i)} e^{\lambda t} \\
 &- B^{(i)} k_1 e^{\lambda t} - (B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3) e^{\lambda t} \\
 &- \lambda e^{\lambda t}\} = 0
 \end{aligned} \tag{252}$$

Divide above equation by $e^{\lambda t}$ and we get

$$\begin{aligned}
 s(\psi B^{(i)} \frac{a}{[a+S^{(i)}]^2} - B^{(i)} k_3) &- v(B^{(i)} k_2 + \gamma B^{(i)}) \\
 &+ b\{\psi f(S^{(i)}) - \gamma V^{(i)} - B^{(i)} k_1 - (B^{(i)} k_1 \\
 &+ V^{(i)} k_2 + S^{(i)} k_3) - \lambda\} = 0
 \end{aligned} \tag{253}$$

We can summary our Chemostat three equations term of s v and b :

$$\begin{aligned}
 s\{[1 - S^{(i)}]k_3 - [B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3] \\
 - \psi \frac{a B^{(i)}}{[a+S^{(i)}]^2} - \lambda\} &+ v\{[1 - S^{(i)}]k_2 + b\{[1 - S^{(i)}]k_1 \\
 &- \psi \frac{S^{(i)} a}{[a+S^{(i)}]^2} - \psi \frac{[S^{(i)}]^2}{[a+S^{(i)}]^2}\} = 0
 \end{aligned} \tag{254}$$

$$\begin{aligned}
 -s V^{(i)} k_3 + v\{\gamma e^{-\lambda \tau} B^{(i)} - \varphi - \gamma B^{(i)} \frac{1}{\beta} \\
 - V^{(i)} k_2 - [B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3] - \lambda\} \\
 + b\{\gamma V^{(i)} e^{-\lambda \tau} - \gamma V^{(i)} \frac{1}{\beta} - V^{(i)} k_1\} = 0
 \end{aligned} \tag{255}$$

$$\begin{aligned}
 s(\psi B^{(i)} \frac{a}{[a+S^{(i)}]^2} - B^{(i)} k_3) &- v(B^{(i)} k_2 + \gamma B^{(i)}) \\
 &+ b\{\psi f(S^{(i)}) - \gamma V^{(i)} - B^{(i)} k_1 - (B^{(i)} k_1 \\
 &+ V^{(i)} k_2 + S^{(i)} k_3) - \lambda\} = 0
 \end{aligned} \tag{256}$$

The small increments Jacobian of our Chemostat system with control variables is as below:

$$\begin{pmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} \\ \Xi_{21} & \Xi_{22} & \Xi_{23} \\ \Xi_{31} & \Xi_{32} & \Xi_{33} \end{pmatrix} \begin{pmatrix} s \\ v \\ b \end{pmatrix} = 0 \tag{257}$$

$$\begin{aligned}
 \Xi_{11} &= [1 - S^{(i)}]k_3 - [B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3] \\
 &- \psi \frac{a B^{(i)}}{[a+S^{(i)}]^2} - \lambda
 \end{aligned} \tag{258}$$

$$\begin{aligned}
 \Xi_{12} &= [1 - S^{(i)}]k_2; \Xi_{13} = [1 - S^{(i)}]k_1 - \psi f(S^{(i)}) \\
 \Xi_{21} &= -V^{(i)} k_3; \Xi_{22} = \gamma e^{-\lambda \tau} B^{(i)} - \varphi - \gamma B^{(i)} \frac{1}{\beta} \\
 &- V^{(i)} k_2 - [B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3] - \lambda
 \end{aligned} \tag{259}$$

$$\begin{aligned}
 \frac{S^{(i)} a}{[a+S^{(i)}]^2} + \frac{[S^{(i)}]^2}{[a+S^{(i)}]^2} &= \frac{S^{(i)} a + [S^{(i)}]^2}{[a+S^{(i)}]^2} \\
 \frac{S^{(i)} a}{[a+S^{(i)}]^2} + \frac{[S^{(i)}]^2}{[a+S^{(i)}]^2} &= \frac{S^{(i)}}{[a+S^{(i)}]} = f(S^{(i)}) \\
 \Xi_{23} &= \gamma V^{(i)} e^{-\lambda \tau} - \gamma V^{(i)} \frac{1}{\beta} - V^{(i)} k_1
 \end{aligned} \tag{260}$$

$$\begin{aligned}
 \Xi_{31} &= \psi B^{(i)} \frac{a}{[a+S^{(i)}]^2} - B^{(i)} k_3 \\
 \Xi_{32} &= -(B^{(i)} k_2 + \gamma B^{(i)})
 \end{aligned} \tag{261}$$

$$\begin{aligned}
 \Xi_{33} &= \psi f(S^{(i)}) - \gamma V^{(i)} - B^{(i)} k_1 \\
 &- (B^{(i)} k_1 + V^{(i)} k_2 + S^{(i)} k_3) - \lambda
 \end{aligned} \tag{262}$$

$$\begin{pmatrix} [1 - S^{(i)}]k_3 & [1 - S^{(i)}]k_1 & \\ -[B^{(i)} k_1 & -\psi \frac{S^{(i)} a}{[a+S^{(i)}]^2} & \\ + V^{(i)} k_2 + S^{(i)} k_3] & [1 - S^{(i)}]k_2 & -\psi \frac{[S^{(i)}]^2}{[a+S^{(i)}]^2} \\ -\psi \frac{a B^{(i)}}{[a+S^{(i)}]^2} - \lambda & & -\psi \frac{[S^{(i)}]^2}{[a+S^{(i)}]^2} \\ & \gamma e^{-\lambda \tau} B^{(i)} & \\ & -\varphi & \\ & -\gamma B^{(i)} \frac{1}{\beta} & \gamma V^{(i)} e^{-\lambda \tau} \\ -V^{(i)} k_3 & -V^{(i)} k_2 & -\gamma V^{(i)} \frac{1}{\beta} \\ & -[B^{(i)} k_1 & -V^{(i)} k_1 \\ & + V^{(i)} k_2 & \\ & + S^{(i)} k_3] & \\ & -\lambda & \\ & & \psi f(S^{(i)}) \\ & & -\gamma V^{(i)} \\ & & -B^{(i)} k_1 \\ & & -B^{(i)} k_1 \\ & & -B^{(i)} k_1 \\ & & + V^{(i)} k_2 \\ & & + S^{(i)} k_3 \\ & & -\lambda \end{pmatrix} \begin{pmatrix} s \\ v \\ b \end{pmatrix} = 0 \tag{263}$$

$$A - \lambda I = \begin{pmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} \\ \Xi_{21} & \Xi_{22} & \Xi_{23} \\ \Xi_{31} & \Xi_{32} & \Xi_{33} \end{pmatrix}; \det|A - \lambda I| = 0$$

$$A = \begin{pmatrix} \Upsilon_{11} & \Upsilon_{12} & \Upsilon_{13} \\ \Upsilon_{21} & \Upsilon_{22} & \Upsilon_{23} \\ \Upsilon_{31} & \Upsilon_{32} & \Upsilon_{33} \end{pmatrix} \tag{264}$$

$$\Upsilon_{11} = [1 - S^{(i)}]k_3 - [B^{(i)}k_1 + V^{(i)}k_2 + S^{(i)}k_3] - \psi \frac{aB^{(i)}}{[a+S^{(i)}]^2}; \Upsilon_{12} = [1 - S^{(i)}]k_2 \quad (265)$$

$$\begin{aligned} \Upsilon_{13} &= [1 - S^{(i)}]k_1 - \psi f(S^{(i)}); \Upsilon_{21} = -V^{(i)}k_3 \\ \Upsilon_{22} &= \gamma e^{-\lambda\tau} B^{(i)} - \varphi - \gamma B^{(i)} \frac{1}{\beta} - V^{(i)}k_2 \\ &- [B^{(i)}k_1 + V^{(i)}k_2 + S^{(i)}k_3] \\ \Upsilon_{23} &= \gamma V^{(i)} e^{-\lambda\tau} - \gamma V^{(i)} \frac{1}{\beta} - V^{(i)}k_1 \end{aligned} \quad (266)$$

$$\begin{aligned} \Upsilon_{31} &= (\psi B^{(i)} \frac{a}{[a+S^{(i)}]^2} - B^{(i)}k_3) \\ \Upsilon_{32} &= -B^{(i)}(k_2 + \gamma) \\ \Upsilon_{33} &= \psi f(S^{(i)}) - \gamma V^{(i)} - B^{(i)}k_1 \\ &- (B^{(i)}k_1 + V^{(i)}k_2 + S^{(i)}k_3) \end{aligned} \quad (267)$$

$$\begin{aligned} \det[A - \lambda I] &= 0 \\ (\Upsilon_{11} - \lambda) \det \begin{pmatrix} (\Upsilon_{22} - \lambda) & \Upsilon_{23} \\ \Upsilon_{32} & (\Upsilon_{33} - \lambda) \end{pmatrix} \\ - \Upsilon_{12} \det \begin{pmatrix} \Upsilon_{21} & \Upsilon_{23} \\ \Upsilon_{31} & (\Upsilon_{33} - \lambda) \end{pmatrix} \\ + \Upsilon_{13} \det \begin{pmatrix} \Upsilon_{21} & (\Upsilon_{22} - \lambda) \\ \Upsilon_{31} & \Upsilon_{32} \end{pmatrix} &= 0 \end{aligned} \quad (268)$$

$$\begin{aligned} (\Upsilon_{11} - \lambda) \{(\Upsilon_{22} - \lambda)(\Upsilon_{33} - \lambda) - \Upsilon_{32}\Upsilon_{23}\} \\ - \Upsilon_{12} \{ \Upsilon_{21}(\Upsilon_{33} - \lambda) - \Upsilon_{31}\Upsilon_{23} \} \\ + \Upsilon_{13} \{ \Upsilon_{21}\Upsilon_{32} - \Upsilon_{31}(\Upsilon_{22} - \lambda) \} &= 0 \end{aligned} \quad (269)$$

$$\begin{aligned} (\Upsilon_{11} - \lambda) \{ \Upsilon_{22}\Upsilon_{33} - \Upsilon_{32}\Upsilon_{23} \\ - \lambda(\Upsilon_{22} + \Upsilon_{33}) + \lambda^2 \} \\ - \Upsilon_{12} \{ \Upsilon_{21}\Upsilon_{33} - \Upsilon_{31}\Upsilon_{23} - \Upsilon_{21}\lambda \} \\ + \Upsilon_{13} \{ \Upsilon_{21}\Upsilon_{32} - \Upsilon_{31}\Upsilon_{22} + \Upsilon_{31}\lambda \} &= 0 \end{aligned} \quad (270)$$

$$\begin{aligned} \Upsilon_{11}\Upsilon_{22}\Upsilon_{33} - \Upsilon_{11}\Upsilon_{32}\Upsilon_{23} \\ - \lambda\Upsilon_{11}(\Upsilon_{22} + \Upsilon_{33}) + \Upsilon_{11}\lambda^2 \\ - \lambda\Upsilon_{22}\Upsilon_{33} + \lambda\Upsilon_{32}\Upsilon_{23} + \lambda^2(\Upsilon_{22} + \Upsilon_{33}) \\ - \lambda^3 - \Upsilon_{12}\Upsilon_{21}\Upsilon_{33} + \Upsilon_{12}\Upsilon_{31}\Upsilon_{23} + \Upsilon_{12}\Upsilon_{21}\lambda \\ + \Upsilon_{13}\Upsilon_{21}\Upsilon_{32} - \Upsilon_{13}\Upsilon_{31}\Upsilon_{22} + \Upsilon_{13}\Upsilon_{31}\lambda &= 0 \end{aligned} \quad (271)$$

$$\begin{aligned} -\lambda^3 + \lambda^2[\Upsilon_{11} + \Upsilon_{22} + \Upsilon_{33}] + \lambda[\Upsilon_{32}\Upsilon_{23} \\ - \Upsilon_{22}\Upsilon_{33} + \Upsilon_{12}\Upsilon_{21} - \Upsilon_{11}(\Upsilon_{22} + \Upsilon_{33}) \\ + \Upsilon_{13}\Upsilon_{31}] + \Upsilon_{11}\Upsilon_{22}\Upsilon_{33} - \Upsilon_{11}\Upsilon_{32}\Upsilon_{23} \\ - \Upsilon_{12}\Upsilon_{21}\Upsilon_{33} + \Upsilon_{12}\Upsilon_{31}\Upsilon_{23} + \Upsilon_{13}\Upsilon_{21}\Upsilon_{32} \\ - \Upsilon_{13}\Upsilon_{31}\Upsilon_{22} &= 0 \end{aligned} \quad (272)$$

We define the following parameters:

$$\begin{aligned} \Xi_3 &= -1; \Xi_2(\tau, k_1, k_2, k_3) = \Upsilon_{11} + \Upsilon_{22} + \Upsilon_{33} \\ \Xi_1(\tau, k_1, k_2, k_3) &= \Upsilon_{32}\Upsilon_{23} - \Upsilon_{22}\Upsilon_{33} + \Upsilon_{12}\Upsilon_{21} \\ &- \Upsilon_{11}(\Upsilon_{22} + \Upsilon_{33}) + \Upsilon_{13}\Upsilon_{31} \end{aligned} \quad (273)$$

$$\begin{aligned} \Xi_0 &= \Upsilon_{11}\Upsilon_{22}\Upsilon_{33} - \Upsilon_{11}\Upsilon_{32}\Upsilon_{23} - \Upsilon_{12}\Upsilon_{21}\Upsilon_{33} \\ &+ \Upsilon_{12}\Upsilon_{31}\Upsilon_{23} + \Upsilon_{13}\Upsilon_{21}\Upsilon_{32} - \Upsilon_{13}\Upsilon_{31}\Upsilon_{22} \end{aligned} \quad (274)$$

We get our Chemostat system with control variables characteristic equation:

$$\begin{aligned} D(\lambda, \tau, k_1, k_2, k_3) &= \sum_{j=0}^3 \lambda^j \Xi_j(\tau, k_1, k_2, k_3) \\ \tau > 0; k_1 > 0, k_2 > 0, k_3 > 0 \end{aligned} \quad (275)$$

We have three Chemostat stability cases:

- (1) $\tau > 0; k_1 = k; k_2 = k_3 = 0$
- (2) $\tau > 0; k_1 = k_2 = k; k_3 = 0$
- (3) $\tau > 0; k_1 = k_2 = k_3 = k$

We need to get characteristics equations for all above stability analysis cases. We study the occurrence of any possible stability switching resulting from the increase of value of the time delay τ and k_1, k_2, k_3 parameters for the general characteristic equation: $D(\lambda, \tau, k_1, k_2, k_3)$ or $D(\lambda, \tau, k)$. If we choose τ and k parameters then the following:

$$D(\lambda, \tau, k) = P_n(\lambda, \tau, k) + Q_m(\lambda, \tau, k)e^{-\lambda\tau} \quad (276)$$

IV. CHEMOSTAT SYSTEM WITH CONTROL PARAMETER AND DELAYED VARIABLES IN TIME CHARACTERISTIC EQUATION

We choose to analyze for first case: $\tau > 0; k_1 = k$ and $k_2 = k_3 = 0$. We need to get characteristics equation:

$$D(\lambda, \tau, k) = P_n(\lambda, \tau, k) + Q_m(\lambda, \tau, k)e^{-\lambda\tau} \quad (277)$$

$$\Upsilon_{11} = -B^{(i)} \{ k + \psi \frac{a}{[a+S^{(i)}]^2} \}; \Upsilon_{12} = 0 \quad (278)$$

$$\begin{aligned} \Upsilon_{13} &= [1 - S^{(i)}]k - \psi f(S^{(i)}); \Upsilon_{21} = 0 \\ \Upsilon_{22} &= B^{(i)}[\gamma e^{-\lambda\tau} - \varphi - \gamma \frac{1}{\beta} - k] \\ \Upsilon_{23} &= \gamma V^{(i)} e^{-\lambda\tau} - \gamma V^{(i)} \frac{1}{\beta} - V^{(i)}k \\ \Upsilon_{23} &= V^{(i)}[\gamma e^{-\lambda\tau} - \gamma \frac{1}{\beta} - k] \end{aligned} \quad (279)$$

$$\begin{aligned} \Upsilon_{31} &= \psi B^{(i)} \frac{a}{[a+S^{(i)}]^2}; \Upsilon_{32} = -B^{(i)}\gamma \\ \Upsilon_{33} &= \psi f(S^{(i)}) - \gamma V^{(i)} - 2B^{(i)}k \end{aligned} \quad (280)$$

$$\begin{aligned} \Xi_3 &= -1; \Xi_2 = \Upsilon_{11} + \Upsilon_{22} + \Upsilon_{33} \\ \Xi_2 &= -B^{(i)}\psi \frac{a}{[a+S^{(i)}]^2} - B^{(i)}\varphi - \gamma B^{(i)} \frac{1}{\beta} \\ &+ \psi f(S^{(i)}) - \gamma V^{(i)} - 4B^{(i)}k + B^{(i)}\gamma e^{-\lambda\tau} \\ \Xi_{21} &= -B^{(i)}\psi \frac{a}{[a+S^{(i)}]^2} - B^{(i)}\varphi - \gamma B^{(i)} \frac{1}{\beta} \\ &+ \psi f(S^{(i)}) - \gamma V^{(i)} - 4B^{(i)}k \\ \Xi_{22} &= \gamma B^{(i)}; \Xi_2 = \Xi_{21} + \Xi_{22}e^{-\lambda\tau} \end{aligned} \quad (281)$$

$$\begin{aligned} \Xi_1 |_{\Upsilon_{12}=0, \Upsilon_{21}=0} &= \Upsilon_{32}\Upsilon_{23} - \Upsilon_{22}\Upsilon_{33} \\ &- \Upsilon_{11}(\Upsilon_{22} + \Upsilon_{33}) + \Upsilon_{13}\Upsilon_{31} \end{aligned} \quad (282)$$

$$\begin{aligned} \Xi_1 &= -B^{(i)}\gamma V^{(i)}[\gamma e^{-\lambda\tau} - \gamma \frac{1}{\beta} - k] \\ &B^{(i)}[\gamma e^{-\lambda\tau} - \varphi - \gamma \frac{1}{\beta} - k][\psi f(S^{(i)}) - \gamma V^{(i)} \\ &- 2B^{(i)}k] + B^{(i)}\{k + \psi \frac{a}{[a+S^{(i)}]^2}\} \{B^{(i)}\gamma e^{-\lambda\tau} \\ &- B^{(i)}(\varphi + \gamma \frac{1}{\beta} + k) + \psi f(S^{(i)}) - \gamma V^{(i)} \\ &- 2B^{(i)}k\} + \{[1 - S^{(i)}]k \\ &- \psi f(S^{(i)})\} \{ \psi B^{(i)} \frac{a}{[a+S^{(i)}]^2} \} \end{aligned} \quad (283)$$

$$\begin{aligned} \Xi_1 &= -B^{(i)}\gamma V^{(i)}\gamma e^{-\lambda\tau} + B^{(i)}\gamma V^{(i)}(\gamma \frac{1}{\beta} + k) \\ &- B^{(i)}[\psi f(S^{(i)}) - \gamma V^{(i)} - 2B^{(i)}k]\gamma e^{-\lambda\tau} \\ &+ B^{(i)}(\varphi + \gamma \frac{1}{\beta} + k)[\psi f(S^{(i)}) - \gamma V^{(i)} - 2B^{(i)}k] \\ &+ B^{(i)}\{k + \psi \frac{a}{[a+S^{(i)}]^2}\} \{B^{(i)}[\gamma e^{-\lambda\tau} \\ &- (\varphi + \gamma \frac{1}{\beta} + k)] + \psi f(S^{(i)}) - \gamma V^{(i)} - 2B^{(i)}k\} \\ &+ \{[1 - S^{(i)}]k - \psi f(S^{(i)})\} \{ \psi B^{(i)} \frac{a}{[a+S^{(i)}]^2} \} \end{aligned} \quad (284)$$

$$\begin{aligned} \Xi_1 = & -B^{(i)}\gamma V^{(i)}\gamma e^{-\lambda\tau} + B^{(i)}\gamma V^{(i)}(\gamma\frac{1}{\beta} + k) \\ & -B^{(i)}[\psi f(S^{(i)}) - \gamma V^{(i)} - 2B^{(i)}k]\gamma e^{-\lambda\tau} \\ & + B^{(i)}(\varphi + \gamma\frac{1}{\beta} + k)[\psi f(S^{(i)}) - \gamma V^{(i)} - 2B^{(i)}k] \\ & + B^{(i)}\{k + \psi\frac{a}{[a+S^{(i)}]^2}\}B^{(i)}\gamma e^{-\lambda\tau} \\ & -B^{(i)}(\varphi + \gamma\frac{1}{\beta} + k) + \psi f(S^{(i)}) - \gamma V^{(i)} \\ & -2B^{(i)}k + \{[1 - S^{(i)}]k - \psi f(S^{(i)})\} \\ & \{\psi B^{(i)}\frac{a}{[a+S^{(i)}]^2}\} \end{aligned} \quad (285)$$

$$\begin{aligned} \Xi_1 = & -B^{(i)}\gamma V^{(i)}\gamma e^{-\lambda\tau} + B^{(i)}\gamma V^{(i)}(\gamma\frac{1}{\beta} + k) \\ & -B^{(i)}[\psi f(S^{(i)}) - \gamma V^{(i)} - 2B^{(i)}k]\gamma e^{-\lambda\tau} \\ & + B^{(i)}(\varphi + \gamma\frac{1}{\beta} + k)[\psi f(S^{(i)}) - \gamma V^{(i)} - 2B^{(i)}k] \\ & + B^{(i)}\{k + \psi\frac{a}{[a+S^{(i)}]^2}\}B^{(i)}\gamma e^{-\lambda\tau} \\ & + B^{(i)}\{k + \psi\frac{a}{[a+S^{(i)}]^2}\}[\psi f(S^{(i)}) - \gamma V^{(i)} \\ & -2B^{(i)}k - B^{(i)}(\varphi + \gamma\frac{1}{\beta} + k)] \\ & + \{[1 - S^{(i)}]k - \psi f(S^{(i)})\}\psi B^{(i)}\frac{a}{[a+S^{(i)}]^2} \end{aligned} \quad (286)$$

$$\begin{aligned} \Xi_{12} = & \{-\gamma V^{(i)} + B^{(i)}\{k + \psi\frac{a}{[a+S^{(i)}]^2}\} \\ & -[\psi f(S^{(i)}) - \gamma V^{(i)} - 2B^{(i)}k]\}B^{(i)}\gamma \end{aligned} \quad (287)$$

$$\Xi_1 = \Xi_{11} + \Xi_{12}e^{-\lambda\tau} \quad (288)$$

$$\begin{aligned} \Xi_{11} = & B^{(i)}\gamma V^{(i)}(\gamma\frac{1}{\beta} + k) + B^{(i)}(\varphi \\ & + \gamma\frac{1}{\beta} + k)[\psi f(S^{(i)}) - \gamma V^{(i)} - 2B^{(i)}k] \\ & + B^{(i)}\{k + \psi\frac{a}{[a+S^{(i)}]^2}\}[\psi f(S^{(i)}) - \gamma V^{(i)} \\ & -2B^{(i)}k - B^{(i)}(\varphi + \gamma\frac{1}{\beta} + k)] \\ & + \{[1 - S^{(i)}]k - \psi f(S^{(i)})\}\psi B^{(i)}\frac{a}{[a+S^{(i)}]^2} \end{aligned} \quad (289)$$

$$\begin{aligned} \Xi_1 = & \{-\gamma V^{(i)} + B^{(i)}\{k + \psi\frac{a}{[a+S^{(i)}]^2}\} - [\psi f(S^{(i)}) \\ & - \gamma V^{(i)} - 2B^{(i)}k]\}B^{(i)}\gamma e^{-\lambda\tau} + B^{(i)}\gamma V^{(i)}(\gamma\frac{1}{\beta} + k) \\ & + B^{(i)}(\varphi + \gamma\frac{1}{\beta} + k)[\psi f(S^{(i)}) - \gamma V^{(i)} - 2B^{(i)}k] \\ & + B^{(i)}\{k + \psi\frac{a}{[a+S^{(i)}]^2}\}[\psi f(S^{(i)}) - \gamma V^{(i)} - 2B^{(i)}k \\ & - B^{(i)}(\varphi + \gamma\frac{1}{\beta} + k)] + \{[1 - S^{(i)}]k - \psi f(S^{(i)})\} \\ & \{\psi B^{(i)}\frac{a}{[a+S^{(i)}]^2}\} \end{aligned} \quad (290)$$

$$\Xi_0|_{\Upsilon_{12}=0, \Upsilon_{21}=0} = \Upsilon_{11}\Upsilon_{22}\Upsilon_{33} - \Upsilon_{11}\Upsilon_{32}\Upsilon_{23} - \Upsilon_{13}\Upsilon_{31}\Upsilon_{22} \quad (291)$$

$$\begin{aligned} \Xi_0|_{\Upsilon_{12}=0, \Upsilon_{21}=0} = & -B^{(i)}\{k + \psi\frac{a}{[a+S^{(i)}]^2}\}B^{(i)}[\gamma e^{-\lambda\tau} \\ & - \varphi - \gamma\frac{1}{\beta} - k][\psi f(S^{(i)}) - \gamma V^{(i)} - 2B^{(i)}k] \\ & - B^{(i)}\{k + \psi\frac{a}{[a+S^{(i)}]^2}\}B^{(i)}\gamma V^{(i)}[\gamma e^{-\lambda\tau} - \gamma\frac{1}{\beta} - k] \\ & - \{[1 - S^{(i)}]k - \psi f(S^{(i)})\}\psi B^{(i)}\frac{a}{[a+S^{(i)}]^2}\{B^{(i)}[\gamma e^{-\lambda\tau} \\ & - \varphi - \gamma\frac{1}{\beta} - k]\} \end{aligned} \quad (292)$$

$$\begin{aligned} \Xi_0|_{\Upsilon_{12}=0, \Upsilon_{21}=0} = & -[B^{(i)}]^2\{k + \psi\frac{a}{[a+S^{(i)}]^2}\}[\psi f(S^{(i)}) \\ & - \gamma V^{(i)} - 2B^{(i)}k][\gamma e^{-\lambda\tau} - (\varphi + \gamma\frac{1}{\beta} + k)] \\ & - [B^{(i)}]^2\{k + \psi\frac{a}{[a+S^{(i)}]^2}\}\gamma V^{(i)}[\gamma e^{-\lambda\tau} - (\gamma\frac{1}{\beta} + k)] \\ & - \{[1 - S^{(i)}]k - \psi f(S^{(i)})\}\psi\frac{a}{[a+S^{(i)}]^2}[B^{(i)}]^2[\gamma e^{-\lambda\tau} \\ & - (\varphi + \gamma\frac{1}{\beta} + k)] \end{aligned} \quad (293)$$

$$\begin{aligned} \Xi_0|_{\Upsilon_{12}=0, \Upsilon_{21}=0} = & -[B^{(i)}]^2\{k \\ & + \psi\frac{a}{[a+S^{(i)}]^2}\}[\psi f(S^{(i)}) \\ & - \gamma V^{(i)} - 2B^{(i)}k]\gamma e^{-\lambda\tau} \\ & + [B^{(i)}]^2\{k + \psi\frac{a}{[a+S^{(i)}]^2}\}[\psi f(S^{(i)}) \\ & - \gamma V^{(i)} - 2B^{(i)}k](\varphi + \gamma\frac{1}{\beta} + k) \\ & - [B^{(i)}]^2\{k + \psi\frac{a}{[a+S^{(i)}]^2}\}\gamma V^{(i)}\gamma e^{-\lambda\tau} \\ & + [B^{(i)}]^2\{k + \psi\frac{a}{[a+S^{(i)}]^2}\}\gamma V^{(i)}(\gamma\frac{1}{\beta} + k) \\ & - \{[1 - S^{(i)}]k - \psi f(S^{(i)})\}\psi\frac{a}{[a+S^{(i)}]^2} \\ & [B^{(i)}]^2\gamma e^{-\lambda\tau} + \{[1 - S^{(i)}]k - \psi f(S^{(i)})\} \\ & \psi\frac{a}{[a+S^{(i)}]^2}[B^{(i)}]^2(\varphi + \gamma\frac{1}{\beta} + k) \end{aligned} \quad (294)$$

$$\begin{aligned} \Xi_0|_{\Upsilon_{12}=0, \Upsilon_{21}=0} = & [B^{(i)}]^2\{k + \psi\frac{a}{[a+S^{(i)}]^2}\}[\psi f(S^{(i)}) \\ & - \gamma V^{(i)} - 2B^{(i)}k](\varphi + \gamma\frac{1}{\beta} + k) + [B^{(i)}]^2\{k \\ & + \psi\frac{a}{[a+S^{(i)}]^2}\}\gamma V^{(i)}(\gamma\frac{1}{\beta} + k) + \{[1 - S^{(i)}]k \\ & - \psi f(S^{(i)})\}\psi\frac{a}{[a+S^{(i)}]^2}[B^{(i)}]^2(\varphi + \gamma\frac{1}{\beta} + k) \\ & - \{[1 - S^{(i)}]k - \psi f(S^{(i)})\}\psi\frac{a}{[a+S^{(i)}]^2} \\ & + \{k + \psi\frac{a}{[a+S^{(i)}]^2}\}[\psi f(S^{(i)}) - \gamma V^{(i)} - 2B^{(i)}k] \\ & + \{k + \psi\frac{a}{[a+S^{(i)}]^2}\}\gamma V^{(i)}[B^{(i)}]^2\gamma e^{-\lambda\tau} \end{aligned} \quad (295)$$

$$\begin{aligned} \Xi_{01}|_{\Upsilon_{12}=0, \Upsilon_{21}=0} = & [B^{(i)}]^2\{k + \psi\frac{a}{[a+S^{(i)}]^2}\} \\ & \{[\psi f(S^{(i)}) - \gamma V^{(i)} - 2B^{(i)}k](\varphi + \gamma\frac{1}{\beta} + k) \\ & + [B^{(i)}]^2\{k + \psi\frac{a}{[a+S^{(i)}]^2}\}\gamma V^{(i)}(\gamma\frac{1}{\beta} + k) \\ & + \{[1 - S^{(i)}]k - \psi f(S^{(i)})\}\psi\frac{a}{[a+S^{(i)}]^2}[B^{(i)}]^2 \\ & (\varphi + \gamma\frac{1}{\beta} + k) \end{aligned} \quad (296)$$

$$\begin{aligned} \Xi_0|_{\Upsilon_{12}=0, \Upsilon_{21}=0} = & \Xi_{01}|_{\Upsilon_{12}=0, \Upsilon_{21}=0} \\ & + \Xi_{02}|_{\Upsilon_{12}=0, \Upsilon_{21}=0}e^{-\lambda\tau} \end{aligned} \quad (297)$$

$$\begin{aligned} \Xi_{02}|_{\Upsilon_{12}=0, \Upsilon_{21}=0} = & -(\{[1 - S^{(i)}]k - \psi f(S^{(i)})\} \\ & \psi\frac{a}{[a+S^{(i)}]^2} + \{k + \psi\frac{a}{[a+S^{(i)}]^2}\}[\psi f(S^{(i)}) - \gamma V^{(i)} \\ & - 2B^{(i)}k] + \{k + \psi\frac{a}{[a+S^{(i)}]^2}\}\gamma V^{(i)})[B^{(i)}]^2\gamma \end{aligned} \quad (298)$$

$$\begin{aligned} D(\lambda, \tau, k) = & \sum_{j=0}^3 \lambda^j \Xi_j(\tau, k) \\ \tau > 0; k_1 = & k > 0, k_2 = 0, k_3 = 0 \end{aligned} \quad (299)$$

$$\begin{aligned} D(\lambda, \tau, k) = & \sum_{j=0}^3 \lambda^j \Xi_j(\tau, k) = \lambda^3 \Xi_3(\tau, k) \\ & + \lambda^2 \Xi_2(\tau, k) + \lambda \Xi_1(\tau, k) + \Xi_0(\tau, k) \end{aligned}$$

$$\begin{aligned} D(\lambda, \tau, k)|_{\Upsilon_{12}=0, \Upsilon_{21}=0} = & -\lambda^3 + \lambda^2(\Xi_{21} \\ & + \Xi_{22}e^{-\lambda\tau}) + \lambda(\Xi_{11} + \Xi_{12}e^{-\lambda\tau}) + \Xi_{01} \\ & + \Xi_{02}e^{-\lambda\tau} \end{aligned} \quad (300)$$

$$\begin{aligned} D(\lambda, \tau, k)|_{\Upsilon_{12}=0, \Upsilon_{21}=0} = & -\lambda^3 + \lambda^2 \Xi_{21} \\ & + \lambda \Xi_{11} + \Xi_{01} + [\lambda \Xi_{12} + \Xi_{02} + \Xi_{22}]e^{-\lambda\tau} \end{aligned} \quad (301)$$

$$\begin{aligned} \Xi_{21} = & \Xi_{21}(k); \Xi_{22}; \Xi_{11} = \Xi_{11}(k); \Xi_{12} = \Xi_{12}(k) \\ \Xi_{01} = & \Xi_{01}(k); \Xi_{02} = \Xi_{02}(k) \end{aligned} \quad (302)$$

$$\begin{aligned} \frac{\partial \Xi_{21}}{\partial k} &= -4B^{(i)}; \frac{\partial \Xi_{22}}{\partial k} = 0; \frac{\partial \Xi_{11}}{\partial k} = B^{(i)} \gamma V^{(i)} \\ &+ B^{(i)}[\psi f(S^{(i)}) - \gamma V^{(i)} - 2B^{(i)}k] - B^{(i)}(\varphi \\ &+ \gamma \frac{1}{\beta} + k)2B^{(i)} + B^{(i)}[\psi f(S^{(i)}) - \gamma V^{(i)} \\ &- 2B^{(i)}k - B^{(i)}(\varphi + \gamma \frac{1}{\beta} + k)] + B^{(i)}\{k \\ &+ \psi \frac{a}{[a+S^{(i)}]^2}\}[-2B^{(i)} - B^{(i)}] + [1 - S^{(i)}] \\ &\{\psi B^{(i)} \frac{a}{[a+S^{(i)}]^2}\} \end{aligned} \quad (303)$$

$$\frac{\partial \Xi_{12}}{\partial k} = 3[B^{(i)}]^2 \gamma \quad (304)$$

$$\begin{aligned} \frac{\partial \Xi_{01}}{\partial k} &= [B^{(i)}]^2[\psi f(S^{(i)}) - \gamma V^{(i)} - 2B^{(i)}k](\varphi + \gamma \frac{1}{\beta} + k) \\ &+ [B^{(i)}]^2\{k + \psi \frac{a}{[a+S^{(i)}]^2}\}(-2B^{(i)})(\varphi + \gamma \frac{1}{\beta} + k) \\ &+ [B^{(i)}]^2\{k + \psi \frac{a}{[a+S^{(i)}]^2}\}[\psi f(S^{(i)}) - \gamma V^{(i)} - 2B^{(i)}k] \\ &+ [B^{(i)}]^2 \gamma V^{(i)}(\gamma \frac{1}{\beta} + k) + [B^{(i)}]^2\{k + \psi \frac{a}{[a+S^{(i)}]^2}\} \gamma V^{(i)} \\ &+ [1 - S^{(i)}]\psi \frac{a}{[a+S^{(i)}]^2}[B^{(i)}]^2(\varphi + \gamma \frac{1}{\beta} + k) \\ &+ \{[1 - S^{(i)}]k - \psi f(S^{(i)})\}\psi \frac{a}{[a+S^{(i)}]^2}[B^{(i)}]^2 \end{aligned} \quad (305)$$

$$\begin{aligned} \frac{\partial \Xi_{02}}{\partial k} &= -([1 - S^{(i)}]\psi \frac{a}{[a+S^{(i)}]^2} + [\psi f(S^{(i)}) \\ &- \gamma V^{(i)} - 2B^{(i)}k] - \{k + \psi \frac{a}{[a+S^{(i)}]^2}\}2B^{(i)} \\ &+ \gamma V^{(i)})[B^{(i)}]^2 \gamma \end{aligned} \quad (306)$$

$$D(\lambda, \tau, k)|_{\tau_{12}=0, \tau_{21}=0} = -\lambda^3 + \lambda^2 \Xi_{21} + \lambda \Xi_{11} + \Xi_{01} + [\lambda \Xi_{12} + \Xi_{02} + \Xi_{22}]e^{-\lambda \tau} \quad (307)$$

$$\begin{aligned} P_n(\lambda, k) &= -\lambda^3 + \lambda^2 \Xi_{21} + \lambda \Xi_{11} + \Xi_{01} \\ Q_m(\lambda, k) &= \lambda \Xi_{12} + (\Xi_{02} + \Xi_{22}) \end{aligned} \quad (308)$$

$$D(\lambda, \tau, k) = P_n(\lambda, k) + Q_m(\lambda, k)e^{-\lambda \tau} \quad (309)$$

$n = 3; m = 1; n > m$

The expression for $P_n(\lambda, k)$ is

$$P_n(\lambda, k) = \sum_{i=0}^{n-3} p_i(k)\lambda^i = p_0(k) + p_1(k)\lambda + p_2(k)\lambda^2 + p_3(k)\lambda^3 \quad (310)$$

$$p_0(k) = \Xi_{01}; p_1(k) = \Xi_{11}; p_2(k) = \Xi_{21}; p_3(k) = -1 \quad (311)$$

The expression for $Q_m(\lambda, k)$ is

$$Q_m(\lambda, k) = \sum_{i=0}^{m-1} q_i(k)\lambda^i = q_0(k) + q_1(k)\lambda \quad (312)$$

$$q_0(k) = \Xi_{02} + \Xi_{22}; q_1(k) = \Xi_{12} \quad (313)$$

The homogeneous system for S, V, B leads to a characteristic equation for the eigenvalue λ having the form

$$P(\lambda, k) + Q(\lambda, k)e^{-\lambda \tau} = 0 \quad (314)$$

$$\begin{aligned} P(\lambda, k) + Q(\lambda, k)e^{-\lambda \tau} &= 0 \\ P(\lambda, k) &= \sum_{i=0}^3 a_i \lambda^i; Q(\lambda, k) = \sum_{i=0}^1 c_i \lambda^i \end{aligned} \quad (315)$$

The coefficients $\{a_i(k), c_i(k)\} \in R$ depend on k (active washout rate balance parameter), other parameters kept as a constant.

$$\begin{aligned} a_0(k) &= \Xi_{01}; a_1(k) = \Xi_{11}; a_2(k) = \Xi_{21} \\ a_3(k) &= -1; c_0(k) = \Xi_{02} + \Xi_{22}; c_1(k) = \Xi_{12} \end{aligned} \quad (316)$$

The designation of the variation argument (k) will subsequently be omitted from P, Q, a_i , and c_i . The coefficients a_i, c_i are continuous, and differential functions of their arguments, and direct substitution shows that $a_0 + c_0 \neq 0$ for $k \in R_+$; that is, $\lambda=0$ is not of the following:

$$\begin{aligned} P(\lambda, k) &= P(\lambda); Q(\lambda, k) = Q(\lambda) \\ P(\lambda, k) + Q(\lambda, k)e^{-\lambda \tau} &= 0 \end{aligned} \quad (317)$$

Furthermore, $P(\lambda), Q(\lambda)$ are analytic functions of λ , for which the following requirements of the analysis [BK] can also be verified in the present case:

- (1) If $\lambda = i\omega, \omega \in \mathbb{R}$, then $P(i\omega) + Q(i\omega) \neq 0$.
- (2) $|Q(\lambda)/P(\lambda)|$ is bounded for $|\lambda| \rightarrow \infty, \text{Re} \lambda \geq 0$. No roots bifurcation from ∞ .
- (3) $F(\omega) = |P(i\omega)|^2 - |Q(i\omega)|^2$ has a finite number of zeros. Indeed, this is a polynomial in ω .
- (4) Each positive root $\omega(k)$ of $F(\omega)=0$ is continuous and differentiable with respect to k parameter.

We assume that $P_n(\lambda, k)$ and $Q_m(\lambda, k)$ cannot have common imaginary roots. That is for any real number ω :

$$P_n(\lambda = i\omega, k) + Q_m(\lambda = i\omega, k) \neq 0 \quad (318)$$

$$\begin{aligned} P_n(\lambda = i\omega, k) &= -\omega^2 \Xi_{21} + \Xi_{01} + i\omega[\Xi_{11} + \omega^2] \\ Q_m(\lambda = i\omega, k) &= i\omega \Xi_{12} + (\Xi_{02} + \Xi_{22}) \end{aligned} \quad (319)$$

$$\begin{aligned} P_n(\lambda = i\omega, k) &= -\omega^2 \Xi_{21} + \Xi_{01} + i\omega[\Xi_{11} + \omega^2] \\ Q_m(\lambda = i\omega, k) &= i\omega \Xi_{12} + (\Xi_{02} + \Xi_{22}) \end{aligned} \quad (320)$$

$$|Q(i\omega, k)|^2 = \omega^2 \Xi_{12}^2 + (\Xi_{02} + \Xi_{22})^2 \quad (321)$$

$$\begin{aligned} F(\omega, k) &= |P(i\omega, k)|^2 - |Q(i\omega, k)|^2 = \omega^6 \\ &+ \omega^4[\Xi_{21}^2 + 2\Xi_{11}] + \omega^2[\Xi_{11}^2 - 2\Xi_{21}\Xi_{01}] + \Xi_{01}^2 \\ &- \omega^2 \Xi_{12}^2 - (\Xi_{02} + \Xi_{22})^2 \end{aligned} \quad (322)$$

$$\begin{aligned} F(\omega, k) &= |P(i\omega, k)|^2 - |Q(i\omega, k)|^2 = \omega^6 \\ &+ \omega^4[\Xi_{21}^2 + 2\Xi_{11}] + \omega^2(\Xi_{11}^2 - 2\Xi_{21}\Xi_{01} - \Xi_{12}^2) \\ &+ \Xi_{01}^2 - (\Xi_{02} + \Xi_{22})^2 \end{aligned} \quad (323)$$

We define the following parameters for simplicity:

$$\begin{aligned} \Phi_0 &= \Xi_{01}^2 - (\Xi_{02} + \Xi_{22})^2 \\ \Phi_2 &= \Xi_{11}^2 - 2\Xi_{21}\Xi_{01} - \Xi_{12}^2 \\ \Phi_4 &= \Xi_{21}^2 + 2\Xi_{11}; \Phi_6 = 1 \end{aligned} \quad (324)$$

Hence $F(\omega, k) = 0$ implies $\sum_{i=0}^3 \Phi_{2i}(k)\omega^{2i} = 0$ and its roots are given by solving the polynomial.

$$\begin{aligned} P_R(i\omega, k) &= -\omega^2 \Xi_{21} + \Xi_{01} \\ P_I(i\omega, k) &= \omega[\Xi_{11} + \omega^2] \\ Q_R(i\omega, k) &= \Xi_{02} + \Xi_{22} \\ Q_I(i\omega, k) &= \omega \Xi_{12} \end{aligned} \quad (325)$$

Hence

$$\sin \theta(\tau) = \frac{-P_R(i\omega, k)Q_I(i\omega, k) + P_I(i\omega, k)Q_R(i\omega, k)}{|Q(i\omega)|^2} \quad (326)$$

$$\cos \theta(\tau) = \frac{P_R(i\omega, k)Q_R(i\omega, k) + P_I(i\omega, k)Q_I(i\omega, k)}{|Q(i\omega)|^2} \quad (327)$$

Which jointly with $F(\omega, k) = 0 \Rightarrow \sum_{l=0}^3 \Phi_{2l}(k)\omega^{2l} = 0$

That is continuous and differentiable in k parameter. We get the following general characteristic equation.

$$\begin{aligned} D(\lambda, \tau, k) &= P_n(\lambda, k) + Q_m(\lambda, k)e^{-\lambda\tau} \\ n &= 3; m = 1; n > m \end{aligned} \quad (328)$$

We use different parameters terminology from our last characteristics parameters definition:

$$p_i(k) \rightarrow a_i; q_i(k) \rightarrow c_i; n = 3; m = 1; n > m \quad (329)$$

Additionally $P_n(\lambda, k) \rightarrow P(\lambda); Q_m(\lambda, k) \rightarrow Q(\lambda)$

$$P(\lambda) = \sum_{l=0}^3 a_l \lambda^l; Q(\lambda) = \sum_{l=0}^1 c_l \lambda^l \quad (330)$$

$$\begin{aligned} P(\lambda) &= -\lambda^3 + \lambda^2 \Xi_{21} + \lambda \Xi_{11} + \Xi_{01} \\ Q(\lambda) &= \lambda \Xi_{12} + (\Xi_{02} + \Xi_{22}) \end{aligned} \quad (331)$$

$n, m \in \mathbb{N}_0; n > m$ and $a_i, c_i : \mathbb{R}_{+0} \rightarrow \mathbb{R}$ are continuous and differentiable function of k such that $a_0 + c_0 \neq 0$. In the following "—" denotes complex and conjugate. $P(\lambda), Q(\lambda)$ are analytic functions in λ and differentiable in k parameter. The coefficients $\{a_i(k, \dots)$ and $c_i(k, \dots)\} \in \mathbb{R}$ depend on Chemostat with control variables system's $k \dots$ values. The designation of the variation arguments (k, \dots) will subsequently be omitted from P, Q, a_i, c_i . The coefficients a_i, c_i are continues and differentiable function of their arguments, and direct substitution shows that $a_0 + c_0 \neq 0; \Xi_{01} + \Xi_{02} + \Xi_{22} \neq 0$. In addition, since the coefficients in P and Q are real, we have $\overline{P(-i\omega)} = P(i\omega)$ and $\overline{Q(-i\omega)} = Q(i\omega)$ thus $\lambda = i\omega, \omega > 0$ may be on eigenvalue of characteristic equation. The analysis consists in identifying the roots of characteristics equation situated on the imaginary axis of the complex λ - plane, whereby increasing the parameters k, \dots $\text{Re} \lambda$ may, at the crossing, change its sign from (-) to (+), that is, from a stable focus E^* to an unstable one, or vice versa. This feature may be further assessed by examining the sign of the partial derivatives with respect to k, \dots , Chemostat system parameters [2] [3].

$$\begin{aligned} \Lambda^{-1}(k) &= \left(\frac{\partial \text{Re} \lambda}{\partial k}\right)_{\lambda=i\omega} \\ \varphi, \gamma, \beta, \psi, \dots &= \text{const}; \omega \in \mathbb{R}_+ \end{aligned} \quad (332)$$

If $\Lambda(k) > 0$ (or < 0), then the crossing proceeds from (-) to (+) (or from (+) to (-)), respectively.

$$\begin{aligned} F(\omega, k) &= |P(i\omega, k)|^2 - |Q(i\omega, k)|^2 \\ F(\omega, k) &= \sum_{l=0}^3 \Phi_{2l}(k)\omega^{2l} \\ F(\omega, k) &= \Phi_0(k) + \Phi_2(k)\omega^2 \\ &+ \Phi_4(k)\omega^4 + \Phi_6(k)\omega^6 \end{aligned} \quad (333)$$

Hence $F(\omega, \tau) = 0$ implies $\sum_{k=0}^4 \Phi_{2k}\omega^{2k} = 0$ When writing $P(\lambda) = P_R(\lambda) + iP_I(\lambda)$ and $Q(\lambda) = Q_R(\lambda) + iQ_I(\lambda)$, and inserting $\lambda = i\omega$ into our system's characteristic equation, ω must satisfy the following:

$$\begin{aligned} \sin \omega\tau &= g(\omega) = \frac{-P_R(i\omega)Q_I(i\omega) + P_I(i\omega)Q_R(i\omega)}{|Q(i\omega)|^2} \\ \cos \omega\tau &= h(\omega) = \frac{P_R(i\omega)Q_R(i\omega) + P_I(i\omega)Q_I(i\omega)}{|Q(i\omega)|^2} \end{aligned} \quad (334)$$

Where $|Q(i\omega)|^2 \neq 0$ in view of requirement (a) above, and $(g, h) \in \mathbb{R}$. Furthermore, it follows above $\sin \omega\tau$ and $\cos \omega\tau$ equations that, by squaring and adding the sides, ω must be a positive root of $F(\omega) = |P(i\omega)|^2 - |Q(i\omega)|^2 = 0$.

$$\begin{aligned} \sin \theta(\tau) &= \frac{-P_R(i\omega)Q_I(i\omega) + P_I(i\omega)Q_R(i\omega)}{|Q(i\omega)|^2} \\ \cos \theta(\tau) &= \frac{P_R(i\omega)Q_R(i\omega) + P_I(i\omega)Q_I(i\omega)}{|Q(i\omega)|^2} \end{aligned} \quad (335)$$

The above equations are identical to those in reference [BK], except that the variation arguments are k parameter or x parameter, instead of τ . $F(\omega)$ is independent of τ ; thus $g(\omega)$ and $h(\omega)$ equations imply $F(\omega) = |P(i\omega)|^2 - |Q(i\omega)|^2 = 0$ equation, but not the other way around. The system of $g(\omega), h(\omega)$, and $F(\omega)=0$ can be solved numerically and graphically over the domain interest in the (k, x) plane. First we solve the polynomial $F(\omega) = |P(i\omega)|^2 - |Q(i\omega)|^2 = 0$, retaining only the real positive roots ω , and discarding the others. The result is a 2D manifold (surface) $\omega = \omega(x, k)$ in a three dimensional (3D) space (x, k, ω) , where ω is continuous and differentiable with respect to its arguments, with the possible exception of infinite derivatives on 1D continuous line. We check which ω 's on the surface satisfy equations $g(\omega) = \dots, h(\omega) = \dots$ for some fixed value of τ . The projection of these lines on the (k, x) plane gives the loci of possible stability transitions of the Chemostat system. In the following, the subscripts $\lambda, \omega, \varphi, \gamma, \beta, k, \dots$ indicate the corresponding partial derivatives. Let us first concentrate on $\Lambda(x)$, remember in $\lambda(\psi, \varphi, \gamma, \beta, k, \dots)$ and $\omega(\psi, \varphi, \gamma, \beta, k, \dots)$, and keeping all parameters except one (x) and τ . The derivation closely follows that in reference [BK]. Differentiating active circulator characteristic equation $P(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0$ with respect to specific parameter (x) , and inverting the derivative, for convenience, one calculates:

Remark: $x = \psi, \varphi, \gamma, \beta, k, \dots, \text{etc}$

$$\begin{aligned} \left(\frac{\partial \Lambda}{\partial x}\right)^{-1} &= \frac{-P_\lambda(\lambda, x)Q(\lambda, x) + Q_\lambda(\lambda, x)P(\lambda, x) - \tau P(\lambda, x)Q(\lambda, x)}{P_x(\lambda, x)Q(\lambda, x) - Q_x(\lambda, x)P(\lambda, x)} \end{aligned} \quad (336)$$

Where $P_\lambda = \frac{\partial P}{\partial \lambda}, \dots$ etc., Substituting $\lambda = i\omega$, and bearing in $P(-i\omega) = P(i\omega), \overline{Q(-i\omega)} = Q(i\omega)$ then $iP_\lambda(i\omega) = P_\omega(i\omega)$ and $iQ_\lambda(i\omega) = Q_\omega(i\omega)$ and that on the surface $|P(i\omega)|^2 = |Q(i\omega)|^2$, one obtains:

$$\left(\frac{\partial \Lambda}{\partial x}\right)^{-1} \Big|_{\lambda=i\omega} = \frac{iP_{\omega}(i\omega, x)P(i\omega, x) + iQ_{\omega}(i\omega, x)Q(\lambda, x) - \tau |P(i\omega, x)|^2}{P_x(i\omega, x)P(i\omega, x) - Q_x(i\omega, x)Q(i\omega, x)} \quad (337)$$

Upon separating into real and imaginary parts, with

$$P = P_R + iP_I; Q = Q_R + iQ_I; P_{\omega} = P_{R\omega} + iP_{I\omega} \quad (338)$$

$$\begin{aligned} Q_{\omega} &= Q_{R\omega} + iQ_{I\omega}; P_x = P_{Rx} + iP_{Ix} \\ Q_x &= Q_{Rx} + iQ_{Ix} \end{aligned} \quad (339)$$

$P^2 = P_I^2 + P_R^2$. When (x) can be k parameter or time delay τ etc., Where for convenience, we have dropped the arguments $(i\omega, x)$, and where

$$F_{\omega} = 2[(P_{R\omega}P_R + P_{I\omega}P_I) - (Q_{R\omega}Q_R + Q_{I\omega}Q_I)] \quad (340)$$

$$\begin{aligned} F_x &= 2[(P_{Rx}P_R + P_{Ix}P_I) - (Q_{Rx}Q_R + Q_{Ix}Q_I)] \\ \omega_x &= -F_x/F_{\omega} \end{aligned} \quad (341)$$

We define U and V :

$$U = (P_R P_{I\omega} - P_I P_{R\omega}) - (Q_R Q_{I\omega} - Q_I Q_{R\omega}) \quad (342)$$

$$V = (P_R P_{Ix} - P_I P_{Rx}) - (Q_R Q_{Ix} - Q_I Q_{Rx}) \quad (343)$$

$$\begin{aligned} F_{\omega} \frac{\partial \omega}{\partial x} + F_x = 0; x \in I \Rightarrow \frac{\partial \omega}{\partial x} = -\frac{F_x}{F_{\omega}} \\ \Lambda^{-1}(x) = \left(\frac{\partial \text{Re} \lambda}{\partial x}\right)_{\lambda=i\omega}; \frac{\partial \omega}{\partial x} = \omega_x = -\frac{F_x}{F_{\omega}} \end{aligned} \quad (344)$$

$$\Lambda(x) = \frac{2P^2}{F_x^2 + 4V^2} F_{\omega} \left(\tau \omega_x + \frac{U \omega_x + V}{P^2} \right) \quad (345)$$

$$\text{sign}\{\Lambda^{-1}(x)\} = \text{sign}\left\{\left(\frac{\partial \text{Re} \lambda}{\partial x}\right)_{\lambda=i\omega}\right\} \quad (346)$$

$$\begin{aligned} \text{sign}\{\Lambda^{-1}(x)\} \\ = \text{sign}\{F_{\omega}\} \text{sign}\left\{\tau \frac{\partial \omega}{\partial x} + \omega + \frac{U \frac{\partial \omega}{\partial x} + V}{|P|^2}\right\} \end{aligned} \quad (347)$$

V. CONCLUSION

A gradostat is a piece of laboratory apparatus for experiments along a nutrient gradient. It is a concentration of chemostats in which the adjacent vessels are connected in both directions. In that article we investigate the dynamical of chemostat which has initial quantities of nutrient, phage and bacteria. A chemostat is a continuous culture device in which microorganisms grow, and submitted to a flow of nutrients. There is an interaction between nutrient, phage and bacteria and it describes by some parameters. Important parameters are decay rate of phage in the vessel (φ), growth rate of bacteria in the vessel (ψ), interaction rate between bacteria and phage in the vessel (γ), average number of free phage released at lysing in the vessel (β), washout rate (D), and talent period (τ). The control parameter in our system is the washout rate (dilution rate D) which is dependent on vessel's microorganism's concentration. We define very

important parameters in our system k_1, k_2 , and k_3 washout rate's balance parameters. The dynamical behaviour of chemostat system with control tool is inspected deeply and stability analysis is done.

VI. APPENDIX A

A. Appendix A₁ (Lemma 1.1)

Assume that $\omega(\tau)$ is a positive and real root of $F(\omega, \tau) = 0$ and defined for $\tau \in I, I \subseteq \mathbb{R}_{+0}$, and at some $\tau^* \in I$, $S_n(\tau^*) = 0$. For some $n \in N_0$ then a pair of simple conjugate pure imaginary roots $\lambda_+(\tau^*) = i\omega(\tau^*)$ and $\lambda_-(\tau^*) = -i\omega(\tau^*)$ of $D(\lambda, \tau) = 0$ exist at $\tau = \tau^*$ which crosses the imaginary axis from left to right if $\delta(\tau^*) > 0$ and cross the imaginary axis from right to left if $\delta(\tau^*) < 0$ where

$$\begin{aligned} \delta(\tau^*) &= \text{sign}\left\{\frac{d \text{Re} \lambda}{d\tau}\right\}_{\lambda=i\omega(\tau^*)} = \\ &= \text{sign}\{F_{\omega}(\omega(\tau^*), \tau^*)\} \text{sign}\left\{\frac{dS_n(\tau)}{d\tau}\right\}_{\tau=\tau^*} \end{aligned} \quad (348)$$

The theorem becomes

$$\begin{aligned} \text{sign}\left\{\frac{d \text{Re} \lambda}{d\tau}\right\}_{\lambda=i\omega \pm} \\ = \text{sign}\{\pm \Delta^{1/2}\} \text{sign}\left\{\frac{dS_n(\tau)}{d\tau}\right\}_{\tau=\tau^*} \end{aligned} \quad (349)$$

C. Appendix A₃ (Theorem 1.3)

The characteristic equation has a pair of simple and conjugate pure imaginary roots $\lambda = \pm i\omega(\tau^*)$, $\omega(\tau^*)$ real at $\tau^* \in I$ if $S_n(\tau^*) = \tau^* - \tau_n(\tau^*) = 0$ for some $n \in N_0$. If $\omega(\tau^*) = \omega_+(\tau^*)$, this pair of simple conjugate pure imaginary roots crosses the imaginary axis from left to right if $\delta_+(\tau^*) > 0$ and crosses the imaginary axis from right to left if $\delta_+(\tau^*) < 0$ where

$$\begin{aligned} \delta_+(\tau^*) &= \text{sign}\left\{\frac{d \text{Re} \lambda}{d\tau}\right\}_{\lambda=i\omega_+(\tau^*)} = \\ &= \text{sign}\left\{\frac{dS_n(\tau)}{d\tau}\right\}_{\tau=\tau^*} \end{aligned} \quad (350)$$

If $\omega(\tau^*) = \omega_-(\tau^*)$, this pair of simple conjugate pure imaginary roots cross the imaginary axis from left to right if $\delta_-(\tau^*) > 0$ and crosses the imaginary axis from right to left if $\delta_-(\tau^*) < 0$ where

$$\begin{aligned} \delta_-(\tau^*) &= \text{sign}\left\{\frac{d \text{Re} \lambda}{d\tau}\right\}_{\lambda=i\omega_-(\tau^*)} = \\ &= -\text{sign}\left\{\frac{dS_n(\tau)}{d\tau}\right\}_{\tau=\tau^*} \end{aligned} \quad (351)$$

If $\omega_+(\tau^*) = \omega_-(\tau^*) = \omega(\tau^*)$ then $\Delta(\tau^*) = 0$ and $\text{sign}\left\{\frac{d \text{Re} \lambda}{d\tau}\right\}_{\lambda=i\omega(\tau^*)} = 0$ and the same is true when $S_n(\tau^*) = 0$. The following result can be useful in identifying values of τ where stability switches happened.

D. Appendix A₄ (Theorem 1.4)

Assume that for all $\tau \in I$, $\omega(\tau)$ is defined as a solution of $F(\omega, \tau) = 0$ then $\delta_{\pm}(\tau) = \text{sign}\{\pm \Delta^{1/2}(\tau)\} \text{sign} D_{\pm}(\tau)$.

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