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Some Properties of Associates of Subsets of FSP-Points Set

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ABSTRACT

In this paper, based upon Fs-set theory [1], we define a crisp Fs-points set FSP($\mathcal{A}$) for given Fs-set $\mathcal{A}$ and establish a pair of relations between collection of all Fs-subsets of a given Fs-set $\mathcal{A}$ and collection of all crisp subsets of Fs-points set FSP($\mathcal{A}$) of the same Fs-set $\mathcal{A}$ and prove one of the relations is a meet complete homomorphism and the other is a join complete homomorphism and search properties of relations between Fs-complemented sets and complemented constructed crisp sets via these homomorphisms.

Key word: Fs-set, Fs-subset, Fs-complement, Fs-function, Fs-point

1 Introduction:

Ever since Zadeh [17] introduced the notion of fuzzy sets in his pioneering work, several mathematicians studied numerous aspects of fuzzy sets.

Murthy[7] introduced f-sets in order to prove Axiom of choice for fuzzy sets. The following example shows why the introduction of f-set theory is necessitated. Let A be non-empty and consider a diamond lattice $\mathrm{L} = \{0, \alpha \parallel \beta, 1\}$. Define two fuzzy sets f and g from A into L such that $\mathrm{f(x)} = \alpha$ and $\mathrm{g(x)} = \beta$. Here both f and g are nonempty fuzzy sets. The Cartesian product of f and g from A into L is given by $(\mathrm{f} \times \mathrm{g})(\mathrm{x}) = \mathrm{f(x)} \wedge \mathrm{g(x)} = \alpha \wedge \beta = 0$. That is, $\mathrm{f} \times \mathrm{g}$ is a empty set. Even though both f and g are non-empty fuzzy sets, their fuzzy Cartesian product is empty showing that the failure of Axiom of choice in L-fuzzy set theory [10]. The collection of all f-subsets of a given f-set with Murthy's definition [7] f-complement [10] could not form a compete Boolean algebra. Vaddiparthi Yogeswara , G.Srinivas and Biswajit Rath introduced the concept of Fs-set and developed the theory of Fs-sets in order to prove collection of all Fs-subsets of given Fs-set is a complete Boolean algebra under Fs-unions, Fs-intersections and Fs-complements. The Fs-sets they introduced contain Boolean valued membership functions .They are successful in their efforts in proving that result with some conditions. In papers [2] and [3] Vaddiparthi Yogeswara, Biswajit Rath and S.V.G.Reddy introduced the concept of Fs-Function between two Fs-subsets of given Fs-set and defined an image of an Fs-subset under a given Fs-function. Also they studied the properties of images under various kinds of Fs-functions.

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In this paper, we construct a crisp set FSP$(\mathcal{A})$ of all Fs-points of given Fs-set $\mathcal{A}$ such that there is a pair of relations between collection of all Fs-subsets of $\mathcal{A}$ and collection of all crisp sub sets of FSP$(\mathcal{A})$,such that one of the relations is a complete meet homomorphism and other is a complete join homomorphism .Here the operations on collection of Fs-subsets of $\mathcal{A}$ are Fs-union, Fs-intersection and Fs-complement. The operations on FSP$(\mathcal{A})$ of are usual crisp set union, crisp set intersection and crisp set complement. The correspondences between them are denoted by the same symbol '~' in the later contexts. The detailed definitions of Fs-point and FSP $(\mathcal{A})$ for given Fs-set $\mathcal{A}$ are discussed before defining those relations mentioned above. For smooth reading of paper, the theory of Fs-sets and Fs-functions in brief is dealt with in first two sections. We denote the largest element of a complete Boolean algebra L_A[1.1] by M_A or 1. We denote Fs-union and crisp set union by same symbol $\cup$ and similarly Fs-intersection and crisp set intersection by the same symbol $\cap$. For all lattice theoretic properties and Boolean algebraic properties one can refer Szasz [13], Garret Birkhoff[14],Steven Givant • Paul Halmos[12] and Thomas Jech[15]

2 Fs-Sets

2.1 Definition

Let U be a universal set, $A_1 \subseteq U$ and let A⊆U be non-empty. A four tuple $\mathcal{A} = \left(A_1, A, \overline{A}\left(\mu_{1A_1}, \mu_{2A}\right), L_A\right)$ is said be an Fs-set if, and only if

(1) $A \subseteq A_1$
(2) L_A is a complete Boolean Algebra
(3) $\mu_{1A_1}: A_1 \longrightarrow L_A$, $\mu_{2A}: A \longrightarrow L_A$,are functions such that $\mu_{1A_1}|A \geq \mu_{2A}$
(4) $\overline{A}$:A⟶ L_A is defined by
$\overline{A}x = \mu_{1A_1}x \wedge (\mu_{2A}x)^c$, for each $x \in A$

2.2 Definition:

Fs-subset

Let $\mathcal{A}=\left(A_1, A, \overline{A}\left(\mu_{1A_1}, \mu_{2A}\right), L_A\right)$ and $\mathcal{B}=\left(B_1, B, \overline{B}\left(\mu_{1B_1}, \mu_{2B}\right), L_B\right)$ be a pair of Fs-sets. $\mathcal{B}$ is said to be an Fs-subset of $\mathcal{A}$, denoted by $\mathcal{B}\subseteq\mathcal{A}$, if, and only if

(1) $B_1 \subseteq A_1$, $A \subseteq B$
(2) L_B is a complete subalgebra of L_A or $L_B \leq L_A$
(3) $\mu_{1B_1} \leq \mu_{1A_1}|B_1$, and $\mu_{2B}|A \geq \mu_{2A}$

2.3 Proposition:

Let $\mathcal{B}$ and $\mathcal{A}$ be a pair of Fs-sets such that $\mathcal{B} \subseteq \mathcal{A}$. Then $\overline{B}x \leq \overline{A}x$ is true for each $x \in A$

2.3.1 Remark:

For some L_X, such that $L_X \leq L_A$ a four tuple $\mathcal{X} = \left(X_1, X, \overline{X}\left(\mu_{1X_1}, \mu_{2X}\right), L_X\right)$ is not an Fs-set if, and only if

(a)$X \nsubseteq X_1$ or

(b) $\mu_{1X_1}x \ngeq \mu_{2X}x$, for some $x \in X \cap X_1$

Here onwards, any object of this type is called an Fs-empty set of first kind and we accept that it is an Fs-subset of $\mathcal{B}$ for any $\mathcal{B} \subseteq \mathcal{A}$.

2.4 Definition:

An Fs-subset $\mathcal{Y}=\left(\mathrm{Y}_1, \mathrm{Y}, \overline{\mathrm{Y}}\left(\mu_{1\mathrm{Y}_1}, \mu_{2\mathrm{Y}}\right), \mathrm{L}_\mathrm{Y}\right)$ of $\mathcal{A}$, is said to be an Fs-empty set of second kind if, and only if

(a') $\mathrm{Y}_1 = \mathrm{Y}$
(b') $\mathrm{L}_\mathrm{Y} \leq \mathrm{L}_\mathrm{A}$
(c') $\overline{\mathrm{Y}} = 0$

2.4.1 Remark:

We denote Fs-empty set of first kind or Fs-empty set of second kind by $\Phi_{\mathcal{A}}$.

2.5 Definition:

Let $\mathcal{B}_1 = \left(\mathrm{B}_{11}, \mathrm{B}_1, \overline{\mathrm{B}}_1\left(\mu_{1\mathrm{B}_{11}}, \mu_{2\mathrm{B}_1}\right), \mathrm{L}_{\mathrm{B}_1}\right)$ and $\mathcal{B}_2 = \left(\mathrm{B}_{12}, \mathrm{B}_2, \overline{\mathrm{B}}_2\left(\mu_{1\mathrm{B}_{12}}, \mu_{2\mathrm{B}_2}\right), \mathrm{L}_{\mathrm{B}_2}\right)$ be a pair of Fs-sets. We say that $\mathcal{B}_1$and $\mathcal{B}_2$ are equal, denoted by $\mathcal{B}_1 = \mathcal{B}_2$ if, only if

(1) $\mathrm{B}_{11} = \mathrm{B}_{12}, \ \mathrm{B}_1 = \mathrm{B}_2$

(2) $\mathrm{L}_{\mathrm{B}_1} = \mathrm{L}_{\mathrm{B}_2}$

(3) (a) $\left(\mu_{1\mathrm{B}_{11}} = \mu_{1\mathrm{B}_{12}}\right.$ and $\left.\mu_{2\mathrm{B}_1} = \mu_{2\mathrm{B}_2}\right)$,or (b) $\overline{\mathrm{B}}_1 = \overline{\mathrm{B}}_2$

2.5.1 Remark:

We can easily observed that 3(a) and 3(b) not equivalent statements.

2.6 Proposition:

$\mathcal{B}_1 = \left(\mathrm{B}_{11}, \mathrm{B}_1, \overline{\mathrm{B}}_1\left(\mu_{1\mathrm{B}_{11}}, \mu_{\mathrm{B}_1}\right), \mathrm{L}_{\mathrm{B}_1}\right)$ and $\mathcal{B}_2 = \left(\mathrm{B}_{12}, \mathrm{B}_2, \overline{\mathrm{B}}_2\left(\mu_{1\mathrm{B}_{12}}, \mu_{\mathrm{B}_2}\right), \mathrm{L}_{\mathrm{B}_2}\right)$ are equal if, only if $\mathcal{B}_1 \subseteq \mathcal{B}_2$ and $\mathcal{B}_2 \subseteq \mathcal{B}_1$

2.7 Definition of Fs-union for a given pair of Fs-subsets of $\mathcal{A}$:

Let $\mathcal{B}=\left(\mathrm{B}_1, \mathrm{B}, \overline{\mathrm{B}}\left(\mu_{1\mathrm{B}_1,}\mu_{2\mathrm{B}}\right), \mathrm{L}_\mathrm{B}\right)$ and

$\mathcal{C}= \left(\mathrm{C}_1, \mathrm{C}, \overline{\mathrm{C}}\left(\mu_{1\mathrm{C}_1}, \mu_{2\mathrm{C}}\right), \mathrm{L}_\mathrm{C}\right)$, be a pair of Fs-subsets of$\mathcal{A}$. Then,

the Fs-union of $\mathcal{B}$ and $\mathcal{C}$,denoted by $\mathcal{B}\cup\mathcal{C}$ is defined as

$\mathcal{B}\cup\mathcal{C}=\mathcal{D}=\left(\mathrm{D}_1, \mathrm{D}, \overline{\mathrm{D}}\left(\mu_{1\mathrm{D}_1}, \mu_{2\mathrm{D}}\right), \mathrm{L}_\mathrm{D}\right)$,where

(1) $\mathrm{D}_1 = \mathrm{B}_1 \cup \mathrm{C}_1 , \ \mathrm{D} = \mathrm{B} \cap \mathrm{C}$
(2) $\mathrm{L}_\mathrm{D} = \mathrm{L}_\mathrm{B} \vee \mathrm{L}_\mathrm{C}$ = complete subalgebra generated by $\mathrm{L}_\mathrm{B} \cup \mathrm{L}_\mathrm{C}$
(3) $\mu_{1\mathrm{D}_1} : \mathrm{D}_1 \longrightarrow \mathrm{L}_\mathrm{D}$ is defined by
$\mu_{1\mathrm{D}_1}\mathrm{x} = (\mu_{1\mathrm{B}_1} \vee \mu_{1\mathrm{C}_1})\mathrm{x}$
$\mu_{2\mathrm{D}} : \mathrm{D} \longrightarrow \mathrm{L}_\mathrm{D}$ is defined by
$\mu_{2\mathrm{D}}\mathrm{x} = \mu_{2\mathrm{B}}\mathrm{x} \wedge \mu_{2\mathrm{C}}\mathrm{x}$
$\overline{\mathrm{D}}: \mathrm{D} \longrightarrow \mathrm{L}_\mathrm{D}$ is defined by
$\overline{\mathrm{D}}\mathrm{x} = \mu_{1\mathrm{D}_1}\mathrm{x} \wedge (\mu_{2\mathrm{D}}\mathrm{x})^c$

2.8 Proposition:

$\mathcal{B}\cup\mathcal{C}$ is an Fs-subset of $\mathcal{A}$.

2.9 Definition of Fs-intersection for a given pair of Fs-subsets of $\mathcal{A}$:

Let $\mathcal{B}=\left(B_1, B, \overline{B}\left(\mu_{1B_1,}\mu_{2B}\right), L_B\right)$ and $\mathcal{C}=\left(C_1, C, \overline{C}\left(\mu_{1C_1}, \mu_{2C}\right), L_C\right)$ be a pair of Fs-subsets of$\mathcal{A}$ satisfying the following conditions:

(i) $B_1 \cap C_1 \supseteq B \cup C$

(ii) $\mu_{1B_1}x \wedge \mu_{1C_1}x \geq (\mu_{2B} \vee \mu_{2C})x$, for each $x \in A$

Then, the Fs-intersection of $\mathcal{B}$ and t, denoted by $\mathcal{B}\cap\mathcal{C}$ is defined as

$\mathcal{B}\cap\mathcal{C}=\mathcal{E}=\left(E_1, E, \overline{E}\left(\mu_{1E_1}, \mu_{2E}\right), L_E\right)$,where

(a) $E_1 = B_1 \cap C_1$, $E = B \cup C$

(b) $L_E = L_B \wedge L_C = L_B \cap L_C$

(c) $\mu_{1E_1}: E_1 \longrightarrow L_E$ is defined by $\mu_{1E_1}x = \mu_{1B_1}x \wedge \mu_{1C_1}x$

$\mu_{2E} : E \longrightarrow L_E$ is defined by

$\mu_{2E}x = (\mu_{2B} \vee \mu_{2C})x$

$\overline{E}: E \longrightarrow L_E$ is defined by

$\overline{E}x = \mu_{1E_1}x \wedge (\mu_{2E}x)^c$.

2.9.1 Remark:

If (i) or (ii) fails we define $\mathcal{B}\cap\mathcal{C}$ as $\mathcal{B}\cap\mathcal{C}=\Phi_{\mathcal{A}}$, which is the Fs-empty set of first kind.

2.10 Proposition:

For any Fs-subsets $\mathcal{B}$, $\mathcal{C}$and $\mathcal{D}$ of $\mathcal{A} = \left(A_1, A, \overline{A}\left(\mu_{1A_1}, \mu_{2A}\right), L_A\right)$, the following associative laws are true:

(I) $\mathcal{B} \cup (\mathcal{C} \cup \mathcal{D}) = (\mathcal{B} \cup \mathcal{C}) \cup \mathcal{D}$

(II) $\mathcal{B} \cap (\mathcal{C} \cap \mathcal{D}) = (\mathcal{B} \cap \mathcal{C}) \cap \mathcal{D}$, whenever Fs-intersections exist.

2.11 Arbitrary Fs-unions and arbitrary Fs-intersections:

Given a family $(\mathcal{B}_i)_{i\in I}$ of Fs-subsets of $\mathcal{A}=\left(A_1, A, \overline{A}\left(\mu_{1A_1}, \mu_{2A}\right), L_A\right)$, where

$\mathcal{B}_i = \left(B_{1i}, B_i, \overline{B}_i\left(\mu_{1B_{1i}}, \mu_{2B_i}\right), L_{B_i}\right)$,for any $i \in I$

2.12 Definition of Fs-union is as follows

Case (1): For I=Φ, define Fs-union of $(\mathcal{B}_i)_{i\in I}$, denoted by $\bigcup_{i\in I}\mathcal{B}_i$ as$\bigcup_{i\in I}\mathcal{B}_i = \Phi_{\mathcal{A}}$, which is the Fs-empty set

Case (2): Define for I≠Φ, Fs-union of $(\mathcal{B}_i)_{i\in I}$ denoted by $\bigcup_{i\in I}\mathcal{B}_i$ as follow

$$\bigcup_{i\in I}\mathcal{B}_i = \mathcal{B} = \left(B_1, B, \overline{B}\left(\mu_{1B_1,}\mu_{2B}\right), L_B\right),$$

where

(a) $B_1 = \bigcup_{i\in I} B_{1i}$,$B = \bigcap_{i\in I} B_i$

(b) $L_B = \bigvee_{i\in I} L_{B_i}$ = complete subalgebra generated by $\bigcup L_i (L_i = L_{B_i})$

(c) $\mu_{1B_1}: B_1 \rightarrow L_B$ is defined by

$\mu_{1B_1}x = \left(\bigvee_{i\in I}\mu_{1B_{1i}}\right)x = \bigvee_{i\in I_x}\mu_{1B_{1i}}x$, where

$I_x = \{i \in I | \ x \in B_i\}$

$\mu_{2B}: B \longrightarrow L_B$ is defined by $\mu_{2B}x = \left(\bigwedge_{i\in I}\mu_{2B_i}\right)x$

$=\bigwedge_{i\in I}\mu_{2B_i}x$

$\overline{B}: B \longrightarrow L_B$ is defined by $\overline{B}x = \mu_{1B_1}x \wedge (\mu_{2B}\, x)^c$

2.12.1 Remark

We can easily show that (d) $B_1 \supseteq B$ and $\mu_{1B_1}|B \geq \mu_{2B}$.

2.13 Definition of Fs-intersection:

Case (1): For I=Φ, we define Fs-intersection of $(\mathcal{B}_i)_{i\in I}$, denoted by $\bigcap_{i\in I}\mathcal{B}_i$ as $\bigcap_{i\in I}\mathcal{B}_i = \mathcal{A}$

Case (2): Suppose $\bigcap_{i\in I} B_{1i} \supseteq \bigcup_{i\in I} B_i$ and $\bigwedge_{i\in I} \mu_{1B_{1i}}|(\bigcup_{i\in I} B_i) \geq \bigvee_{i\in I} \mu_{2B_i}$

Then, we define Fs-intersection of $(\mathcal{B}_i)_{i\in I}$, denoted by $\bigcap_{i\in I}\mathcal{B}_i$ as follows

$$\bigcap_{i\in I}\mathcal{B}_i = \mathcal{C} = \left(C_1, C, \overline{C}\left(\mu_{1C_1}, \mu_{2C}\right), L_C\right)$$

(a') $C_1 = \bigcap_{i\in I} B_{1i}\, , C = \bigcup_{i\in I} B_i$
(b') $L_C = \bigwedge_{i\in I} L_{B_i}$
(c') $\mu_{1C_1}: C_1 \longrightarrow L_C$ is defined by $\mu_{1C_1}x = \left(\bigwedge_{i\in I}\mu_{1B_{1i}}\right)x = \bigwedge_{i\in I}\mu_{1B_{1i}}\, x$
$\mu_{2C}: C \longrightarrow L_C$ is defined by $\mu_{2C}x = \left(\bigvee_{i\in I}\mu_{2B_i}\right)x = \bigvee_{i\in I_x}\mu_{2B_i}x$,
where, $I_x = \{i \in I |\ x \in B_i\}$
$\overline{C}: C \longrightarrow L_C$ is defined by $\overline{C}x = \mu_{1C_1}x \wedge (\mu_{2C}\, x)^c$

Case (3): $\bigcap_{i\in I} B_{1i} \not\supseteq \bigcup_{i\in I} B_i$ or $\bigwedge_{i\in I} \mu_{1B_{1i}}|(\bigcup_{i\in I} B_i) \ngeq \bigvee_{i\in I} \mu_{2B_i}$

We define

$$\bigcap_{i\in I}\mathcal{B}_i = \Phi_{\mathcal{A}}$$

2.13.1 Lemma:

For any Fs-subset $\mathcal{B} = \left(B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B\right)$ and $\mathcal{B} \subseteq \mathcal{B}_i = \left(B_{1i}, B_i, \overline{B}_i(\mu_{1B_{1i}}, \mu_{2B_i}), L_{B_i}\right)$ for each $i \in I$. $\bigcap_{i\in I}\mathcal{B}_i$ exists and $\mathcal{B} \subseteq \bigcap_{i\in I}\mathcal{B}_i$

2.14 Proposition:

$(\mathcal{L}(\mathcal{A}), \cap)$ is $\wedge$-complete lattics.

2.14.1 Corollary:

For any Fs-subset $\mathcal{B}$ of $\mathcal{A}$, the following results are true

(i) $\Phi_{\mathcal{A}} \cup \mathcal{B} = \mathcal{B}$
(ii) $\Phi_{\mathcal{A}} \cap \mathcal{B} = \Phi_{\mathcal{A}}$.

2.15 Proposition:

$(\mathcal{L}(\mathcal{A}), \cup)$ is $\vee$-complete lattics.

2.15.1 Corollary:

$(\mathcal{L}(\mathcal{A}), \cup, \cap)$ is a complete lattice with $\vee$ and $\wedge$

2.16 Proposition:

Let $\mathcal{B}=\left(B_1, B, \overline{B}\left(\mu_{1B_1}, \mu_{2B}\right), L_B\right)$, $\mathcal{C}=\left(C_1, C, \overline{C}\left(\mu_{1C_1}, \mu_{2C}\right), L_C\right)$ and $\mathcal{D}=\left(D_1, D, \overline{D}\left(\mu_{1D_1}, \mu_{2D}\right), L_D\right)$. Then $\mathcal{B} \cup (\mathcal{C} \cap \mathcal{D})=(\mathcal{B} \cup \mathcal{C}) \cap (\mathcal{B} \cup \mathcal{D})$ provided $\mathcal{C} \cap \mathcal{D}$ exists.

2.17 Proposition:

Let $\mathcal{B}=\left(B_1, B, \overline{B}\left(\mu_{1B_1}, \mu_{2B}\right), L_B\right)$, $\mathcal{C}=\left(C_1, C, \overline{C}\left(\mu_{1C_1}, \mu_{2C}\right), L_C\right)$ and $\mathcal{D}=\left(D_1, D, \overline{D}\left(\mu_{1D_1}, \mu_{2D}\right), L_D\right)$. Then $\mathcal{B} \cap (\mathcal{C} \cup \mathcal{D})=(\mathcal{B} \cap \mathcal{C}) \cup (\mathcal{B} \cap \mathcal{D})$ provided in R.H.S

$(\mathcal{B} \cap \mathcal{C})$ and $(\mathcal{B} \cap \mathcal{D})$ exists.

2.18 Definition of Fs-complement of an Fs-subset:

Consider a particular Fs-set $\mathcal{A} = \left(A_1, A, \overline{A}\,(\mu_{1A_1}, \mu_{2A}), L_A\right)$, $A \neq \Phi$,where

(i) $A \subseteq A_1$
(ii) $L_A = [0, M_A]$, $M_A = \vee\, \overline{A}A = \vee_{a\in A} \overline{A}\, a$
(iii) $\mu_{1A_1} = M_A$, $\mu_{2A} = 0$,
$\overline{A}x = \mu_{1A_1} x \wedge (\mu_{2A} x)^c = M_A$, for each $x \in A$

Given $\mathcal{B}=\left(B_1, B, \overline{B}\left(\mu_{1B_1}, \mu_{2B}\right), L_B\right)$. We define Fs-complement of $\mathcal{B}$, denoted by $\mathcal{B}^{C_{\mathcal{A}}}$ for B=A and $L_B = L_A$ as follows:

$\mathcal{B}^{C_{\mathcal{A}}} = \mathcal{D}=\left(D_1, D, \overline{D}\left(\mu_{1D_1}, \mu_{2D}\right), L_D\right)$, where

(a') $D_1 = C_A B_1 = B_1^c \cup A, D = B = A$
(b') $L_D = L_A$
(c') $\mu_{1D_1}: D_1 \longrightarrow L_A$, is defined by $\mu_{1D_1} x = M_A$
$\mu_{2D}: A \longrightarrow L_A$, is defined by $\mu_{2D} x = \overline{B}x = \mu_{1B_1} x \wedge (\mu_{2B} x)^c$
$\overline{D}: A \longrightarrow L_A$,is defined by $\overline{D}x = \mu_{1D_1} x \wedge (\mu_{2D} x)^c = M_A \wedge (\overline{B}x)^c = (\overline{B}x)^c$.

2.19 Proposition:

$\mathcal{A}^{C_{\mathcal{A}}} = \Phi_{\mathcal{A}}$

2.20 Definition:

Define $(\Phi_{\mathcal{A}})^{C_{\mathcal{A}}} = \mathcal{A}$

2.21 Proposition:

For $\mathcal{B}=\left(B_1, B, \overline{B}\left(\mu_{1B_1}, \mu_{2B}\right), L_B\right)$, $\mathcal{C}= \left(C_1, C, \overline{C}\left(\mu_{1C_1}, \mu_{2C}\right), L_C\right)$, which are non Fs-empty sets and $B = C = A, L_B = L_C = L_A$

(1) $\mathcal{B} \cap \mathcal{B}^{C_{\mathcal{A}}} = \Phi_{\mathcal{A}}$
(2) $\mathcal{B} \cup \mathcal{B}^{C_{\mathcal{A}}} = \mathcal{A}$
(3) $\left(\mathcal{B}^{C_{\mathcal{A}}}\right)^{C_{\mathcal{A}}} = \mathcal{B}$
(4) $\mathcal{B} \subseteq \mathcal{C}$ if and only if $\mathcal{C}^{C_{\mathcal{A}}} \subseteq \mathcal{B}^{C_{\mathcal{A}}}$

2.22 Proposition:

Fs-De-Morgan’s laws for a given pair of Fs-subsets:

For any pair of Fs-sets $\mathcal{B}=\left(B_1, B, \bar{B}\left(\mu_{1B_1,}\mu_{2B}\right), L_B\right)$ and $\mathcal{C}= \left(C_1, C, \bar{C}\left(\mu_{1C_1}, \mu_{2C}\right), L_C\right)$, with $B = C = A$ and $L_B = L_C = L_A$, we will have

(i) $(\mathcal{B} \cup \mathcal{C})^{C_{\mathcal{A}}} = \mathcal{B}^{C_{\mathcal{A}}} \cap \mathcal{C}^{C_{\mathcal{A}}}$ if $(\bar{B}x)^c \wedge (\bar{C}x)^c \leq \left[\left(\mu_{1B_1}x\right)^c \vee \mu_{2C}x\right] \wedge \left[\left(\mu_{1C_1}x\right)^c \vee \mu_{2B}x\right]$, for each $x \in A$

(ii)$(\mathcal{B} \cap \mathcal{C})^{C_{\mathcal{A}}} = \mathcal{B}^{C} \cup \mathcal{C}^{C_{\mathcal{A}}}$, whenever $\mathcal{B} \cap \mathcal{C}$ exists.

2.23 Fs-De Morgan laws for any given arbitrary family of Fs-sets:

Proposition: Given a family of Fs-subsets $(\mathcal{B}_i)_{i\in I}$ of $\mathcal{A} = \left(A_1, A, \bar{A}\left(\mu_{1A_1}, \mu_{2A}\right), L_A\right)$,where $L_A=[0,M_A]$.$\mu_{1A_1} = M_A, \mu_{2A} = 0, \bar{A}x = M_A$

(I) $(\cup_{i\in I}\mathcal{B}_i)^{C_{\mathcal{A}}} = \cap_{i\in I}\mathcal{B}_i^{C_{\mathcal{A}}}$, for I≠Φ, where$\mathcal{B}_i = \left(B_{1i}, B_i, \bar{B}_i\left(\mu_{1B_{1i}}, \mu_{2B_i}\right), L_{B_i}\right)$ and

(1) $B_i = A, L_{B_i} = L_A$ provided $\wedge_{i\in I}(\bar{B}_i x)^c \leq \wedge_{\substack{i,j\in I\\ i\neq j}}\left[\left(\mu_{1B_{1i}}x\right)^c \vee \mu_{2B_j}x\right]$

(II) $(\cap_{i\in I}\mathcal{B}_i)^{C_{\mathcal{A}}} = \cup_{i\in I}\mathcal{B}_i^{C_{\mathcal{A}}}$, whenever $\cap_{i\in I}\mathcal{B}_i$ exist

3 Fs-point

3.1 Definition

We define an object, for $b \in A, \beta \in L_A$ such that $\beta \leq \bar{A}b$ - denoted by (b, β) as follows

$(b, \beta) = \left(B_1, B, \bar{B}\left(\mu_{1B_1}, \mu_{2B}\right), L_B\right)$, where $A \subseteq B \subseteq B_1 \subseteq A_1, L_B \leq L_A$, such that $\mu_{1B_1}x, \mu_{2B}x \in L_B$, $\alpha \leq \mu_{1A_1}x, \forall x \in A_1, \beta \in L_A$

$$\mu_{1B_1}x = \begin{cases} \mu_{2A}x, & x \neq b, x \in A \\ \beta \vee \mu_{2A}b, & x = b \\ \alpha, & x \notin A, x \in A_1 \end{cases} \quad \text{And } \mu_{2B}x = \begin{cases} \mu_{2A}x, & x \in A \\ \alpha, & x \notin A, x \in B \end{cases}$$

3.2 Lemma:

(a) $\beta \leq \mu_{1A_1}b$ and $\beta \leq (\mu_{2A}b)^c$
(b) $\mu_{1B_1}b \geq \mu_{2B}b$
(c) $\mu_{1B_1}b \leq \mu_{1A_1}b$
(d) $\mu_{2B}b \geq \mu_{2A}b$
(e) $\bar{B}b = \beta$
(f) (b, β) is Fs-subset of $\mathcal{A}$

Proof : (a): Given $\beta \leq \bar{A}b = \mu_{1A_1}b \wedge (\mu_{2A}b)^c$

$\Rightarrow \beta \leq \mu_{1A_1}b$ and $\beta \leq (\mu_{2A}b)^c$

(b): $\mu_{1B_1}b \wedge \mu_{2B}b = (\beta \vee \mu_{2A}b) \wedge \mu_{2A}b = \mu_{2A}b = \mu_{2B}b$

$\Rightarrow \mu_{1B_1}b \geq \mu_{2B}b$

(c): $\mu_{1B_1}b \wedge \mu_{1A_1}b = (\beta \vee \mu_{2A}b) \wedge \mu_{1A_1}b = \left(\beta \wedge \mu_{1A_1}b\right) \vee \left(\mu_{1A_1}b \wedge \mu_{2A}b\right) = \beta \vee \mu_{2A}b = \mu_{1B_1}b$

$\Rightarrow \mu_{1B_1}b \leq \mu_{1A_1}b$

(d): $\mu_{2B}b \geq \mu_{2A}b$ ($\because \mu_{2B}x = \mu_{2A}x, \forall x \in A$)

(e): $\bar{B}b = \mu_{1B_1}b \wedge (\mu_{2B}b)^c$

$= (\beta \vee \mu_{2A}b) \wedge (\mu_{2A}b)^c$

$= (\beta \wedge (\mu_{2A}b)^c) \vee (\mu_{2A}b \wedge (\mu_{2A}b)^c)$

$= (\beta \wedge (\mu_{2A} b)^c) \vee 0$

$= \beta \wedge (\mu_{2A} b)^c = \beta$

(f): Given $(b, \beta) = (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B)$

(i) $B_1 \subseteq A_1, A \subseteq B$

(ii) $L_B \leq L_A$

(iii) $\mu_{1B_1} x = \begin{cases} \mu_{2A} x, & x \neq b, x \in A \\ \beta \vee \mu_{2A} b, & x = b \\ \alpha, & x \notin A, x \in A_1 \end{cases}$ And $\mu_{2B} x = \begin{cases} \mu_{2A} x, & x \in A \\ \alpha, & x \notin A, x \in B \end{cases}$

$\mu_{1B_1} x = \mu_{2A} x = \mu_{2B} x \leq \mu_{1A_1} x, x \neq b, x \in A$

$\mu_{1B_1} b = \beta \vee \mu_{2A} b \geq \mu_{2A} b = \mu_{2B} b, \mu_{1B_1} b \leq \mu_{1A_1} b$

$\therefore \mu_{1B_1} x \geq \mu_{2B} x, \forall x \in B, \mu_{1B_1} x \leq \mu_{1A_1} x, \forall x \in B_1$ and $\mu_{2B} x = \mu_{2A} x, \forall x \in A$

Hence $(b, \beta) = (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B)$ is Fs-subset of $\mathcal{A}$.

Here onward (b, β) —which is an Fs-subset of $\mathcal{A}$, we call a (b, β) objects of $\mathcal{A}$.

3.3 Definition of a relation between objects:

For any (b, β) objects $\mathcal{B}_1 = (B_{11}, B_1, \overline{B}_1(\mu_{1B_{11}}, \mu_{2B_1}), L_{B_1})$ and $\mathcal{B}_2 = (B_{12}, B_2, \overline{B}_2(\mu_{1B_{12}}, \mu_{2B_2}), L_{B_2})$ of $\mathcal{A}$, we say that $\mathcal{B}_1 R(b, \beta) \mathcal{B}_2$ if, and only if $\mu_{1B_{11}} x = \mu_{2B_1} x, x \neq b$ and $\forall x \in B_1$ and $\mu_{1B_{12}} x = \mu_{2B_2} x, x \neq b$ and $\forall x \in B_2$ and $\mu_{1B_{11}} b = \mu_{1B_{12}} b = \beta \vee \mu_{2A} b$ and $\mu_{2B_1} b = \mu_{2B_2} b = \mu_{2A} b$.

3.4 Theorem:

$R(b, \beta)$ is an equivalence relation.

Proof: The proof follows clearly from definition.

3.5 Definition of Fs-point:

The equivalence class corresponding to $R(b, \beta)$ is denoted by χ_b^β or (b, β).We define this χ_b^β is an Fs point of $\mathcal{A}$.

Set of all Fs-point of $\mathcal{A}$ is denoted by FSP($\mathcal{A}$).

3.6 Definition:

Let $G \subseteq \mathrm{FSP}(\mathcal{A})$.

(a) G is said to be closed under stalks if, and only if $\chi_b^\beta \in G, \alpha \leq \beta \Rightarrow \chi_b^\alpha \in G$

(b) G is said to be closed under supremums if and only if $M \subseteq L_A, \chi_b^\beta \in G, \forall \beta \in M \Rightarrow \chi_b^{\vee M} \in G$, $\vee M = \bigvee_{\beta \in M} \beta$

(c) G is said to be S-closed if, and only if G is closed under both stalks and supremums.

3.7 Theorem:

Arbitrary intersection of S-closed subset is S-closed

3.8 Definition:

Let $G \subseteq \mathrm{FSP}(\mathcal{A})$.

Define $G^{\sim} = \Phi_{\mathcal{A}}$ if $G = \Phi$.Otherwise $G^{\sim} = \cup_{\chi_b^\beta \in G} \chi_b^\beta$

Define $\mathcal{B} = \left(B_1, B, \overline{B}\left(\mu_{1B_1}, \mu_{2B}\right), L_B\right)$, where

$B_1 \supseteq B = \left\{b|\chi_b^\beta \in G\right\}, L_B = \vee_{\chi_b^\beta \in G} L_\beta, \mu_{1B_1}b = \vee_{\chi_b^\beta \in G}(\beta \vee \mu_{2A}b), \mu_{2B}b = \mu_{2A}b$

$\overline{B}b = \mu_{1B_1}b \wedge (\mu_{2B}b)^c$

$= \vee_{\chi_b^\beta \in G}(\beta \vee \mu_{2A}b) \wedge (\mu_{2A}b)^c$

$= \left[\left(\vee_{\chi_b^\beta \in G} \beta\right) \vee \mu_{2A}b\right] \wedge (\mu_{2A}b)^c$

$= \left(\left(\vee_{\chi_b^\beta \in G} \beta\right) \wedge (\mu_{2A}b)^c\right) \vee (\mu_{2A}b \wedge (\mu_{2A}b)^c)$

$= \vee_{\chi_b^\beta \in G}(\beta \wedge (\mu_{2A}b)^c) \vee 0$

$= \vee_{\chi_b^\beta \in G}(\beta \wedge (\mu_{2A}b)^c) = \vee_{\chi_b^\beta \in G} \beta$

3.9 Theorem:

$G^{\sim} = \mathcal{B}$

Proof: Let $\chi_b^\beta = \left(B_1, B, \overline{X}\left(\mu_{1X_1}, \mu_{2X}\right), L_X\right)$, where $\beta \leq \overline{A}b$

$\mu_{1X_1}x = \begin{cases} \mu_{2A}x, & x \neq b, x \in A \\ \beta \vee \mu_{2A}b, & x = b \\ \alpha, & x \notin A, x \in B_1 \end{cases}$ And $\mu_{2X}x = \begin{cases} \mu_{2A}x, & x \in A \\ \alpha, & x \notin A, x \in B \end{cases}$

Let$\cup_{\chi_b^\beta \in G} \chi_b^\beta = \mathcal{C} = \left(C_1, C, \overline{C}\left(\mu_{1C_1}, \mu_{2C}\right), L_C\right)$, where

(I) $C_1 = B_1, C = B = \left\{b|\chi_b^\beta \in G\right\}, C_1 \supseteq C$

(II) $L_C = L_X = \vee_{\chi_b^\beta \in G} L_\beta$

(III) For $b \in A$, $\mu_{1C_1}b = \vee_{\chi_b^\beta \in G}(\beta \vee \mu_{2A}b) = \mu_{1B_1}b, \mu_{2C}b = \mu_{2A}b = \mu_{2B}b$

Hence $G^{\sim} = \mathcal{B}$

3.10 Definition:

For any $\mathcal{B} \subseteq \mathcal{A}$

Define $\mathcal{B}^{\sim} = \Phi$ if $\mathcal{B} = \Phi_{\mathcal{A}}$

Let $\mathcal{B} = \left(B_1, B, \overline{B}\left(\mu_{1B_1}, \mu_{2B}\right), L_B\right)$ and $\mathcal{B} \neq \Phi_{\mathcal{A}}$

Define $\mathcal{B}^{\sim} = \left\{\chi_b^\beta | b \in B, \beta \in L_B, \beta \leq \overline{B}b\right\}$

3.11 Theorem:

$\mathcal{A} = \cup_{\chi_b^\beta \in FSP(\mathcal{A})} \chi_b^\beta$

Proof: The proof follows similar lines of the proof of 4.9

3.12 Lemma:

$\mathcal{A}^{\sim} = \mathrm{FSP}(\mathcal{A})$

Clearly $\mathcal{A}^{\sim} \subseteq \mathrm{FSP}(\mathcal{A})$

Let $\chi_b^{\beta} = \mathcal{B} = \left(B_1, B, \overline{B}\left(\mu_{1B_1}, \mu_{2B}\right), L_B\right) \in \mathrm{FSP}(\mathcal{A})$

$\therefore \chi_b^{\beta}$ is a (b, β) object

I.e. $b \in A, \beta \in L_A, A \subseteq B \subseteq B_1 \subseteq A_1$, such that $\mu_{1B_1}x, \mu_{2B}x \in L_B, L_B \leq L_A, \alpha \leq \mu_{1A_1}x, \forall x \in A_1, \beta \in L_A$

$$\mu_{1B_1}x = \begin{cases} \mu_{2A}x, & x \neq b, x \in A \\ \beta \vee \mu_{2A}b, & x = b \\ \alpha, & x \notin A, x \in A_1 \end{cases} \quad \text{And } \mu_{2B}x = \begin{cases} \mu_{2A}x, & x \in A \\ \alpha, & x \notin A, x \in B \end{cases}$$

Clearly $b \in A \subseteq B, \beta \in L_B$ and $L_B \leq L_A$

Hence $\mathrm{FSP}(\mathcal{A}) \subseteq \mathcal{A}^{\sim}$

Hence $\mathcal{A}^{\sim} = \mathrm{FSP}(\mathcal{A})$

3.13 Theorem:

$\mathcal{B}^{\sim}$ is S-closed.

Proof: Let $\chi_b^{\beta} \in \mathcal{B}^{\sim}$, then $b \in B, \beta \in L_B, \beta \leq \overline{B}b$

Let $\delta \leq \beta, \delta \in L_B, \delta \leq \overline{B}b$

$\therefore \chi_b^{\delta} \in \mathcal{B}^{\sim}$

Hence $\mathcal{B}^{\sim}$ is closed under stalks.

Let $\chi_b^{\beta_i} \in \mathcal{B}^{\sim}$ for $i \in I$ then $b \in B, \beta_i \in L_B, \beta_i \leq \overline{B}b$

$\Rightarrow b \in B, \vee_{i \in I} \beta_i \in L_B, \vee_{i \in I} \beta_i \leq \overline{B}b$

$\Rightarrow \chi_b^{\vee \beta_i} \in \mathcal{B}^{\sim}$

Hence $\mathcal{B}^{\sim}$ is closed under supremum.

$\therefore \mathcal{B}^{\sim}$ is S-closed.

3.14 Theorem:

For any $G \subseteq \mathrm{FSP}(\mathcal{A})$, $G \subseteq G^{\sim\sim}$

Proof: Case (I): $G = \Phi \Rightarrow$ Clear

Case (II): $G \neq \Phi$, we have $G^{\sim\sim} = \mathcal{B}^{\sim} = \left\{\chi_b^{\beta} | b \in B, \beta \in L_B, \beta \leq \overline{B}b\right\}$

Where $\mathcal{B} = \left(B_1, B, \overline{B}\left(\mu_{1B_1}, \mu_{2B}\right), L_B\right)$, where

$B_1 \supseteq B = \left\{b | \chi_b^{\beta} \in G\right\}, L_B = \vee_{\chi_b^{\beta} \in G} L_\beta, \mu_{1B_1}b = \vee_{\chi_b^{\beta} \in G}(\beta \vee \mu_{2A}b), \mu_{2B}b = \mu_{2A}b, \overline{B}b = \vee_{\chi_b^{\beta} \in G} \beta$

Let $\chi_b^{\beta} \in G \Rightarrow b \in B, \beta \in L_B, \overline{B}b = \vee_{\chi_b^{\beta} \in G} \beta, \beta \leq \overline{B}b \Rightarrow \chi_b^{\beta} \in \mathcal{B}^{\sim} = G^{\sim\sim} \Rightarrow G \subseteq G^{\sim\sim}$

3.15 Theorem:

Let $\mathcal{A}$ be an Fs-set. Then the following are equivalent for any $G \subseteq FSP(\mathcal{A})$

(a) $G^{\sim\sim} = G$
(b) G is S-closed
(c) (i) $b \in B \Rightarrow \chi_b^{\bar{B}b} \in G$
(ii) $b \in B, \beta \leq \bar{B}b \Rightarrow \chi_b^{\beta} \in G$ where $\mathcal{B} = G^{\sim}$

Proof: We have $G^{\sim} = \cup_{\chi_b^{\beta} \in G} \chi_b^{\beta}$ and from 4.9, $G^{\sim} = \mathcal{B} = \left(B_1, B, \bar{B}\left(\mu_{1B_1}, \mu_{2B}\right), L_B\right)$, where

$B_1 \supseteq B = \left\{b | \chi_b^{\beta} \in G\right\}, L_B = \vee_{\chi_b^{\beta} \in G} L_{\beta}, \mu_{1B_1}b = \vee_{\chi_b^{\beta} \in G}(\beta \vee \mu_{2A}b), \mu_{2B}b = \mu_{2A}b, \bar{B}b = \vee_{\chi_b^{\beta} \in G} \beta$

Also we have $G^{\sim\sim} = \mathcal{B}^{\sim} = \left\{\chi_b^{\beta} | b \in B, \beta \in L_B, \beta \leq \bar{B}b\right\}$

$G = G^{\sim\sim} = \mathcal{B}^{\sim}$ which is S-closed from 4.13 gives (a) $\Rightarrow$ (b)

(i) Here $G^{\sim} = \mathcal{B} = \left(B_1, B, \bar{B}\left(\mu_{1B_1}, \mu_{2B}\right), L_B\right)$, where

$B_1 \supseteq B = \left\{b | \chi_b^{\beta} \in G\right\}, L_B = \vee_{\chi_b^{\beta} \in G} L_{\beta}, \mu_{1B_1}b = \vee_{\chi_b^{\beta} \in G}(\beta \vee \mu_{2A}b), \mu_{2B}b = \mu_{2A}b, \bar{B}b = \vee_{\chi_b^{\beta} \in G} \beta$

For any $b \in B, \chi_b^{\bar{B}b} \in G$ follows from G is closed under supremums.

(ii) For any $b \in B, \beta \leq \bar{B}b$, we have $\chi_b^{\beta} \in G$, because $\chi_b^{\bar{B}b} \in G$ and G is S-closed which gives (b) $\Rightarrow$ (c).

(c) $\Rightarrow$ (a) is clear.

3.16 Theorem:

For any $\mathcal{B}_1$ and $\mathcal{B}_2$ such that $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \mathcal{A}$, ${\mathcal{B}_1}^{\sim} \subseteq {\mathcal{B}_2}^{\sim}$ provided $B_1 = B_2$ where $\mathcal{B}_1 = \left(B_{11}, B_1, \bar{B}_1\left(\mu_{1B_{11}}, \mu_{2B_1}\right), L_{B_1}\right)$ and $\mathcal{B}_2 = \left(B_{12}, B_2, \bar{B}_2\left(\mu_{1B_{12}}, \mu_{2B_2}\right), L_{B_2}\right)$

Proof: From hypotheses, we have

(1) $B_{11} \subseteq B_{12}, B_1 \supseteq B_2$
(2) $L_{B_1} \leq L_{B_2}$
(3) $\mu_{1B_{11}} \leq \mu_{1B_{12}} | B_{11}, \mu_{2B_1} | B_2 \geq \mu_{2B_2}$

$\chi_b^{\beta} \in {\mathcal{B}_1}^{\sim}$

Implies $b \in B_1, \beta \in L_{B_1}, \beta \leq \bar{B}_1 b$ which implies

$b \in B_2, \beta \in L_{B_2}, \beta \leq \bar{B}_1 b \leq \bar{B}_2 b$ ($\therefore \mathcal{B}_1 \subseteq \mathcal{B}_2$)

so that $\chi_b^{\beta} \in {\mathcal{B}_2}^{\sim}$

3.16.1 Corollary:

$\mathcal{B} \subseteq \mathcal{A} \Longrightarrow FSP(\mathcal{B}) \subseteq FSP(\mathcal{A})$

3.17 Result:

$\mathcal{B}_1 \subseteq \mathcal{B}_2$ implies $\mathcal{B}_1 \subseteq \mathcal{B}_2 \cup \mathcal{B}_3$ for any Fs-subset $\mathcal{B}_3$

3.18 Result:

$\chi_b^{\beta} \subseteq G^{\sim}$ for any $\chi_b^{\beta} \in G$ such that $G \subseteq FSP(\mathcal{A})$.

Proof: $\chi_b^\beta \in G$ is an Fs-point of $\mathcal{A}$ and $G^\sim = \bigcup_{\chi_b^\beta \in G} \chi_b^\beta$ gives $\chi_b^\beta \subseteq G^\sim$

3.19 Recall 1.16 for any Family $(\mathcal{G}_i)_{i\in I}$ of Fs-subsets of $\mathcal{A}$ such that $\mathcal{G}_i \subseteq \mathcal{G}$, $\bigcup_{i\in I}\mathcal{G}_i \subseteq \mathcal{G}$.

3.20 Proposition:

${G_1}^\sim \subseteq {G_2}^\sim$ for any two subsets $G_1$and G_2 of FSP($\mathcal{A}$), such that $G_1 \subseteq G_2$.

Proof: The proof follows clearly if $G_1 = \Phi$ because ${G_1}^\sim = \Phi_{\mathcal{A}}$which is an Fs-empty subset of $\mathcal{A}$

For $G_1 \neq \Phi$,

take $\chi_b^\beta \in G_1 \subseteq G_2$ implying $\chi_b^\beta \subseteq {G_2}^\sim$ from4.18 again implying

$\bigcup_{\chi_b^\beta \in G_1} \chi_b^\beta \subseteq {G_2}^\sim$ from 4.19 so that ${G_1}^\sim \subseteq {G_2}^\sim$

3.21 Theorem:

For any Fs-subset $\mathcal{B}$ of an Fs-set $\mathcal{A}$, $\mathcal{B}^{\sim\sim} = \mathcal{B}$.

Proof: For $\mathcal{B} = \Phi_{\mathcal{A}}$-Fs-empty subset of $\mathcal{A}$, $\mathcal{B}^\sim = \Phi$ the crisp empty subset, $\mathcal{B}^{\sim\sim} = \Phi_{\mathcal{A}} = \mathcal{B}$

For $\mathcal{B} \neq \Phi_{\mathcal{A}}$, $\mathcal{B}^\sim = \left\{\chi_b^\beta | b \in B, \beta \in L_B, \beta \leq \overline{B}b \right\}$.

Since each χ_b^β in $\mathcal{B}^\sim$ is an Fs-subset of $\mathcal{B}$ it follows $\bigcup_{\chi_b^\beta \in \mathcal{B}^\sim} \chi_b^\beta \subseteq \mathcal{B}$.

But $G^\sim = \bigcup_{\chi_b^\beta \in G} \chi_b^\beta$implies $(\mathcal{B}^\sim)^\sim = \bigcup_{\chi_b^\beta \in G^\sim} \chi_b^\beta$

Where $\mathcal{B}^\sim = \left\{\chi_b^\beta | \chi_b^\beta \subseteq \mathcal{B} \right\}$

$\therefore \bigcup_{\chi_b^\beta \in \mathcal{B}^\sim} \chi_b^\beta \subseteq \mathcal{B} = \bigcup_{\chi_b^\beta \in G} \chi_b^\beta = \bigcup_{\chi_b^\beta \in \mathcal{B}^\sim} \chi_b^\beta = (\mathcal{B}^\sim)^\sim$

So that $\mathcal{B}^{\sim\sim} = \mathcal{B}$.

3.22 Theorem:

$(\mathcal{B} \cap \mathcal{C})^\sim = \mathcal{B}^\sim \cap \mathcal{C}^\sim$for any Fs-subsets $\mathcal{B} = \left(B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B\right)$ and $\mathcal{C} = \left(C_1, C, \overline{C}(\mu_{1C_1}, \mu_{2C}), L_C\right)$ of $\mathcal{A}$ such that $B = C$.

Proof: For $\mathcal{B} \cap \mathcal{C} = \Phi_{\mathcal{A}}$, $(\mathcal{B} \cap \mathcal{C})^\sim = (\Phi_{\mathcal{A}})^\sim = \Phi$ which is the crisp empty set

For $\chi_b^\beta \in \mathcal{B}^\sim \cap \mathcal{C}^\sim$, $\chi_b^\beta \subseteq \mathcal{B}$ and $\chi_b^\beta \subseteq \mathcal{C}$ which imply $\chi_b^\beta \subseteq \mathcal{B} \cap \mathcal{C}$ again implying $\chi_b^\beta \in (\mathcal{B} \cap \mathcal{C})^\sim$ -a contradiction

For $\mathcal{B} \cap \mathcal{C} \neq \Phi_{\mathcal{A}}$

Say $\mathcal{B} \cap \mathcal{C} = \mathcal{D} = \left(D_1, D, \overline{D}(\mu_{1D_1}, \mu_{2D}), L_D\right)$, where $D = B = C$

Then $(\mathcal{B} \cap \mathcal{C})^\sim \subseteq \mathcal{B}^\sim$ and $(\mathcal{B} \cap \mathcal{C})^\sim \subseteq \mathcal{C}^\sim$ from 4.16

Implying $(\mathcal{B} \cap \mathcal{C})^\sim \subseteq \mathcal{B}^\sim \cap \mathcal{C}^\sim$

For $\chi_b^\beta \in \mathcal{B}^\sim \cap \mathcal{C}^\sim$

$b \in B, \beta \in L_B, \beta \leq \overline{B}b$ and $b \in C, \beta \in L_C, \beta \leq \overline{C}b$

Implying $b \in B \cap C, \beta \in L_B \cap L_C, \beta \leq \overline{B}b \wedge \overline{C}b = (\overline{B} \wedge \overline{C})b$ again implying $\chi_b^\beta \in (\mathcal{B} \cap \mathcal{C})^\sim$

So that $(\mathcal{B} \cap \mathcal{C})^{\sim} \supseteq \mathcal{B}^{\sim} \cap \mathcal{C}^{\sim}$

Hence $(\mathcal{B} \cap \mathcal{C})^{\sim} = \mathcal{B}^{\sim} \cap \mathcal{C}^{\sim}$

3.23 Proposition:

For any family of Fs-subset $(\mathcal{B}_i)_{i\in I}$ of $\mathcal{A}$, $(\bigcap_{i\in I} \mathcal{B}_i)^{\sim} = \bigcap_{i\in I} \mathcal{B}_i{}^{\sim}$ provided all B_i's are equal for each $i \in I$

3.24 Theorem:

$(G_1 \cup G_2)^{\sim} = G_1{}^{\sim} \cup G_2{}^{\sim}$ for any subsets G_1 and G_2 of FSP$(\mathcal{A})$,

Proof: For $G_1 = \Phi$, we have $G_1{}^{\sim} = \Phi_{\mathcal{A}}$ and $(G_1 \cup G_2)^{\sim} = G_2{}^{\sim}$ and $G_1{}^{\sim} \cup G_2{}^{\sim} = G_2{}^{\sim}$

So that $(G_1 \cup G_2)^{\sim} = G_1{}^{\sim} \cup G_2{}^{\sim}$.

Suppose G_1 and G_2 be non-empty

Since $G_1, G_2 \subseteq G_1 \cup G_2, G_1{}^{\sim}, G_2{}^{\sim} \subseteq (G_1 \cup G_2)^{\sim}$ so that $G_1{}^{\sim} \cup G_2{}^{\sim} \subseteq (G_1 \cup G_2)^{\sim}$

For $\chi_b^{\beta} \subseteq (G_1 \cup G_2)^{\sim}$, $\chi_b^{\beta} \in G_1 \cup G_2$ so that $\chi_b^{\beta} \in G_1$ or $\chi_b^{\beta} \in G_2$ implying $\chi_b^{\beta} \subseteq G_1^{\sim}$ or $\chi_b^{\beta} \subseteq G_2^{\sim}$ so that $\chi_b^{\beta} \subseteq G_1^{\sim} \cup G_2^{\sim}$ finally $(G_1 \cup G_2)^{\sim} \subseteq G_1^{\sim} \cup G_2^{\sim}$

Hence $(G_1 \cup G_2)^{\sim} = G_1{}^{\sim} \cup G_2{}^{\sim}$

3.25 Theorem:

$(\bigcup_{i\in I} G_i)^{\sim} = \bigcup_{i\in I} G_i{}^{\sim}$ for any family $(G_i)_{i\in I}$ of subsets of FSP$(\mathcal{A})$.

3.25.1 Remark:

Observe that χ_c^0 is always an Fs-subset of $\mathcal{B}$ i.e. $\chi_c^0 \in \mathcal{B}^{\sim}$ i.e. $\chi_c^0 \notin (\mathcal{B}^{\sim})^c$

3.26 Theorem:

For $\mathcal{B} = \left(B_1, B, \overline{B}\left(\mu_{1B_1}, \mu_{2B}\right), L_B\right) \subseteq \mathcal{A}, B = A$ and $L_A = L_B$,

$$\left(\mathcal{B}^{\mathcal{C}_{\mathcal{A}}}\right)^{\sim} \subseteq (\mathcal{B}^{\sim})^c$$

Proof: Suppose $\mathcal{B}^{C_{\mathcal{A}}} = \mathcal{D}$=$\left(D_1, D, \overline{D}\left(\mu_{1D_1}, \mu_{2D}\right), L_D\right)$. From 1.18

(1) $D_1 = C_A B_1 = B_1^c \cup A, D = B = A$
(2) $L_D = L_A$
(3) $\mu_{1D_1}: D_1 \longrightarrow L_A$, is defined by $\mu_{1D_1}x = M_A$
$\mu_{2D}: A \longrightarrow L_A$, is defined by $\mu_{2D}x = \overline{B}x = \mu_{1B_1}x \wedge (\mu_{2B}x)^c$
$\overline{D}: A \longrightarrow L_A$,is defined by $\overline{D}x = \mu_{1D_1}x \wedge (\mu_{2D}x)^c = M_A \wedge (\overline{B}x)^c = (\overline{B}x)^c$.

Then from 4.10 $\left(\mathcal{B}^{\mathcal{C}_{\mathcal{A}}}\right)^{\sim} = \mathcal{D}^{\sim} = \left\{\chi_d^{\delta} | d \in A, \delta \in L_A = L_D, \delta \leq \overline{D}d = (\overline{B}d)^c \text{ i.e. } \delta \wedge \overline{B}d = 0\right\}$

And $\mathcal{B}^{\sim} = \left\{\chi_b^{\beta} | \ b \in B = A, \beta \in L_B = L_A, \beta \leq \overline{B}b \right\}$ implying $(\mathcal{B}^{\sim})^c = \left\{\chi_c^{\gamma} | \chi_c^{\gamma} \notin \mathcal{B}^{\sim}\right\}$

$\chi_c^{\gamma} \in \left(\mathcal{B}^{\mathcal{C}_{\mathcal{A}}}\right)^{\sim}$ implying $\gamma \wedge \overline{B}c = 0$ which implies $\gamma \nleq \overline{B}c$ as $\chi_c^{\gamma} \notin \mathcal{B}^{\sim} \Rightarrow \gamma \nleq \overline{B}c$ so that $\chi_c^{\gamma} \in (\mathcal{B}^{\sim})^c$

Hence $\left(\mathcal{B}^{\mathcal{C}_{\mathcal{A}}}\right)^{\sim} \subseteq (\mathcal{B}^{\sim})^c$

3.26.1 Example:

Let $\mathcal{A} = \left(A_1, A, \overline{A}\left(\mu_{1A_1}, \mu_{2A}\right), L_A\right)$, where $A_1 = \{a, b\}, A = \{a\}, \mu_{1A_1} = 1, \mu_{2A} = 0$ and $L_A = \{0, \alpha \parallel \beta, 1\}$

Suppose $\mathcal{B} = \left(B_1, B, \overline{B}\left(\mu_{1B_1}, \mu_{2B}\right), L_B\right) \subseteq \mathcal{A}$, where $B_1 = B = A = \{a\}, \mu_{1B_1} = \alpha, \mu_{2B} = 0$ and $L_B = L_A$ $\overline{B}a = \alpha$

$\mathcal{B}^{C_{\mathcal{A}}} = \mathcal{D}=(D_1, D, \overline{D}(\mu_{1D_1}, \mu_{2D}), L_D)$,where $D_1 = A_1, D = A, \mu_{1D_1}a = 1, \mu_{2D}a = \alpha, \overline{D} = \beta, L_D = L_A$

$(\mathcal{B}^{C_{\mathcal{A}}})^{\sim} = \mathcal{D}^{\sim} = \{\chi_d^{\delta} | d \in A, \delta \in L_A = L_B, \delta \leq \overline{D}d = (\overline{B}d)^c \text{ i.e } \delta \wedge \overline{B}d = 0\}$

$\mathcal{B}^{\sim} = \left\{\chi_b^{\beta} | \ b \in B = A, \beta \in L_B = L_A, \beta \leq \overline{B}b \right\}$

$\Rightarrow \mathcal{B}^{\sim} = \{\chi_a^0, \chi_a^{\alpha}\}$

$\Rightarrow \chi_a^1 \in (\mathcal{B}^{\sim})^c \ (\because 1 \nleq \overline{B}a = \alpha)$

But $\chi_a^1 \notin (\mathcal{B}^{C_{\mathcal{A}}})^{\sim}$

I.e. $(\mathcal{B}^{\sim})^c \nsubseteq (\mathcal{B}^{C_{\mathcal{A}}})$

3.27 Theorem:

$(G^{\sim})^{C_{\mathcal{A}}} \subseteq (G^c)^{\sim}$ for any $G \subseteq FSP(\mathcal{A})$, where $\mathcal{A} = (A_1, A, \overline{A}(\mu_{1A_1}, \mu_{2A}), L_A), \mu_{1A_1} = M_A, \mu_{2A} = 0$ and $L_A = [0, M_A]$.

Proof: For any $G \subseteq FSP(\mathcal{A})$, $G^{\sim} = \bigcup_{\chi_b^{\beta} \in G} \chi_b^{\beta}$

Let $G^{\sim} = \mathcal{B} = (B_1, B, \overline{B}(\mu_{1B_1}, \mu_{2B}), L_B)$, where $B_1 \supseteq B = A = \{b | \chi_b^{\beta} \in G\}, L_B = L_A, \mu_{1B_1}b = \bigvee_{\chi_b^{\beta} \in G}(\beta \vee \mu_{2A}b) = \bigvee_{\chi_b^{\beta} \in G} \beta, \mu_{2B}b = \mu_{2A}b = 0, \ \overline{B}b = \bigvee_{\chi_b^{\beta} \in G} \beta$.

Let $(\mathcal{B})^{C_{\mathcal{A}}} = \mathcal{D} = \mathcal{D} = (D_1, D, \overline{D}(\mu_{1D_1}, \mu_{2D}), L_D)$ then

(1) $D_1 = A, D = B = A$
(2) $L_D = L_A$
(3) $\mu_{1D_1}: D_1 \longrightarrow L_A$, is defined by $\mu_{1D_1}x = M_A$
$\mu_{2D}: A \longrightarrow L_A$, is defined by $\mu_{2D}b = \overline{B}b = \bigvee_{\chi_b^{\beta} \in G} \beta$

$\overline{D}: A \longrightarrow L_A$,is defined by $\overline{D}b = \mu_{1D_1}b \wedge (\mu_{2D}b)^c = M_A \wedge (\overline{B}b)^c = \left(\bigvee_{\chi_b^{\beta} \in G} \beta\right)^c$

Also $(G^{\sim})^{C_{\mathcal{A}}} = (\mathcal{B})^{C_{\mathcal{A}}} = \mathcal{D}$

Now, $G^c = FSP(\mathcal{A}) - G$

Let $(G^c)^{\sim} = \mathcal{E} = (E_1, E, \overline{E}(\mu_{1E_1}, \mu_{2E}), L_E)$, where

$E_1 = E = A = \{c | \chi_c^{\gamma} \in G^c\}, L_E = L_A, \mu_{1E_1}c = \bigvee_{\chi_c^{\gamma} \in G^c}(\gamma \vee \mu_{2A}c) = \bigvee_{\chi_c^{\gamma} \in G^c} \gamma, \mu_{2E}c = \mu_{2A}c = 0, \quad \overline{E}c = \bigvee_{\chi_c^{\gamma} \in G^c} \gamma$.

We prove $(G^{\sim})^{C_{\mathcal{A}}} \subseteq (G^c)^{\sim}$ if $\chi_b^{M_A} \in G$ or $\chi_b^{M_A} \notin G$

If $\chi_b^{M_A} \in G$, then $G^{\sim} = \bigcup_{\chi_b^{\beta} \in G} \chi_b^{\beta} = \mathcal{B} = (B_1, B, \overline{B}(M_A, 0), L_A), \overline{B} = M_A$ implying

$(G^{\sim})^{C_{\mathcal{A}}} = \mathcal{D} = (D_1, D, \overline{D}(M_A, M_A), L_A), \overline{D} = 0$

That is, $(G^{\sim})^{C_{\mathcal{A}}} = \Phi_{\mathcal{A}} \subseteq (G^c)^{\sim}$

If $\chi_b^{M_A} \notin G$ then $\chi_b^{M_A} \in G^c$implying $(G^c)^{\sim} = \mathcal{E} = (E_1, E, \overline{E}(M_A, 0), L_A)$

That is, $(G^c)^{\sim} = \mathcal{A} \supseteq (G^{\sim})^{C_{\mathcal{A}}}$

Hence, whether $\chi_b^{M_A} \in G$ or $\chi_b^{M_A} \notin G$,we have

$(G^{\sim})^{C_{\mathcal{A}}} \subseteq (G^{c})^{\sim}$

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