Advances in Social Sciences Research Journal – Vol.6, No.2
Publication Date: Feb. 25, 2019
DoI:10.14738/assrj.62.6200.
Copertari, L. F. (2019). The complexity coefficient for acyclic networks measuring network density: Application to project management. *Advances in Social Sciences Research Journal,* 6(2) 319-326.

The complexity coefficient for acyclic networks measuring network density: Application to project management

Luis F. Copertari
Computer Engineering Program
Autonomous University of Zacatecas (UAZ). Zacatecas, México

ABSTRACT
A complexity coefficient applied to acyclic networks for project management measuring network density, not its structure, is presented, analyzed and discussed. Both the Activity on Arc (AOA) and the Activity on Node (AON) paradigms are considered. Based on the restriction that the project network needs to be acyclic, it is possible to define strict bounds to the complexity coefficient. In this case, the complexity coefficient, named *eta*, varies between zero and one, and measures only network density.

Keywords: Activity networks; project management; CPM; acyclic; complexity.

INTRODUCTION

Activity networks in project management need to satisfy the requirement that they are not cyclic. That is, they do not have feedback loops. In practice, that means it is possible to number nodes in such way that all arrows always go from a lower numbered node to a higher numbered node. This requirement is independent of the way in which the activities are portrayed, being that Activity on Arc (AOA) or Activity on Node (AON).

The concept of network complexity tends to refer to the number of nodes compared to the number of alternative paths that exist in a given network. The emphasis tends to be towards characterizing the network in terms not only of its density, but also on its structure (Emmert-Streib & Dehmer, 2012; Rigterink & Singer, 2014; Zenil, Kiani, & Tegnér, 2018). Nevertheless, Kaimann defines the Coefficient of Network Complexity (CNC) based on the structural complexity of any given network (Kaimann,1974). In some cases, the complexity coefficient is aimed at measuring properties of chemical structures (Dehmer, Barbarini, Varmuza, & Graber, 2009; Keller, Berger, Liepelt, & Lipowsky, 2013).

The complexity coefficient presented in this paper intuitively measures network density regardless of other structure properties such as randomness. It provides a single coefficient called *eta* (η) that indicates with a value between zero and one (which can also be translated into a percentage for network complexity), how fully connected the network is. Such kind of approach to complexity measurement is novel, because it does not consider network properties, only its density. Also, the coefficient is restricted to the case of acyclic networks in project management.

PROJECT COMPLEXITY

The complexity of a project is measured by the complexity of the activity network used to represent the activity precedence relationships for any given project. The network complexity parameter *eta* (η) indicates how complex is any given project. A value for η equal to zero indicates that the project is minimally connected, that is, it has the minimum possible number of precedence relationships that actually connects all activities in the project. A value for η equal to one indicates that the project is fully connected, that is, it has the maximum number of allowed precedence relationships such that they do not violate the acyclicity of the activity network. Keep in mind that valid activity networks are acyclic, that is, they do not contain feedback loops in the network.

The equations for calculating η greatly depend on the way the activity network is represented. It is possible to use an Activity on Arc (AOA) notation in which activities are denoted using the arcs and the nodes simply indicate a precedence relationship between activities. However, the AOA notation is not suitable for practical algorithmic representation, because of the need (sometimes) to use *dummy* activities. Dummy activities are artificial activities that have zero duration time and are merely used in order not to violate precedence relationships in an AOA notation. Thus, the Activity on Node (AON) representation is used instead. AON networks are such that the nodes indicate the activities and the arrows or arcs indicate precedence relationships between activities. AON networks are easier to implement.

The adjacency matrix (Elsayed & Boucher, 1994) can be used not only to represent the precedence relationships of any given project, but also to check project network acyclicity (or cyclicity). The adjacency matrix in an AOA representation uses nodes to indicate activity precedence and ones in the matrix to indicate the existence of an activity such that node (row) i indicates that node i precedes node (column) j. Conversely, an adjacency matrix in AON notation uses rows and columns to denote actual activities and the existence of a one for activity (row) i and activity (column) j indicate that activity i precedes activity j.

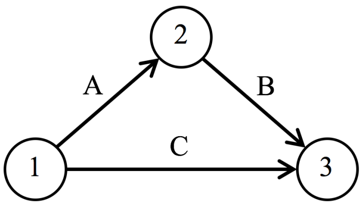
ACTIVITY ON ARC (AOA) NETWORK COMPLEXITY

Let us first explore the issue of acyclicity in AOA networks. Figure 1a shows an acyclic AOA network with three activities, whereas Figure 1b illustrates a cyclic AOA network also having three activities. The inquisitive reader can check that the adjacency matrix of the acyclic network (Figure 1a) allows a complete row/column cancellation, whereas the adjacency matrix of the cyclic network (Figure 1b) does not allow a single row/column cancellation.

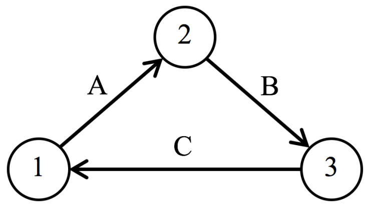
If some rows/columns cannot be cancelled, at least some of the activity network contains cyclicity. The nodes are numbered from 1 to 3 and the activities are called A, B and C. Notice that it is also possible to call activities not only with a name such as A or B, but also with a number, such as 1 or 2. We use numbers for nodes and letters for activities to make it easier not to confuse them.

Figure 1: AOA network representations of acyclic and cyclic illustrative sample projects.

a. Acyclic AOA project network.



b. Cyclic AOA project network.



Let n denote the total number of nodes in any given AOA network and N be the total number of activities in such AOA network. Notice in Figure 1a that for all activities, i < j, that is, all activities start from a lower numbered node and end in a higher numbered node. Conversely, in Figure 1b that is not true, since activity C start in node 3 and end in node 1. There is no suitable arrangement, no matter how much the numbers of the nodes are changed (as long as they are all consecutive), such that all activities satisfy the condition that i < j, where i = 1, …, n-1 and j = 2, …, n for Figure 1b. Thus, the activity network in Figure 1b is cyclic. This condition is very important in order to have valid (acyclic) activity networks.

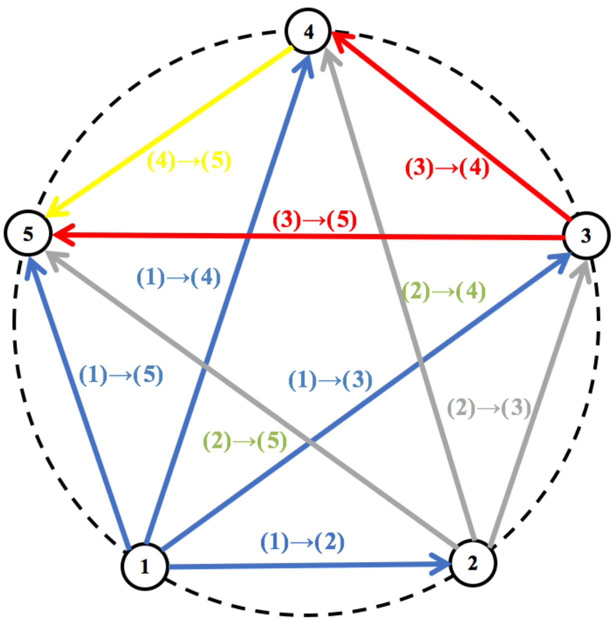
The minimum number of activities in a network with n nodes is n-1, since that is the only way in which all activities are linked in the simplest sequence. Equation (1) denotes such minimum.

$$Min(N) = n-1 \qquad (1)$$

What is the maximum number of activities for an acyclic activity network having n nodes? In order to answer such question, it is useful to use an example. Suppose an activity is denoted as (i)→(j), indicating that the activity starts from node i and ends in node j, where it must always be the case that i < j. Figure 2 shows an activity network having AOA notation of 5 nodes fully connected. Table 1 illustrates the adjacency matrix for the activity network of Figure 2, with the exception that row 5 and column 1 are not shown for not having any ones (precedence relationships). Notice that a full adjacency matrix would have row n (n = 5 in this example) and column 1, which would actually allow to cancel all rows/columns, thus demonstrating that this fully connected activity network is acyclic.

Observe that all nodes can be accommodated around a circle having equidistant distances between them. In this case, n = 5. The first node can be connected with the second, third, fourth and fifth, that is, with n-1 = 5-1 = 4 nodes. The second node can be connected with the third, the fourth and the fifth, that is n-2 = 5-2 = 3 nodes. The third node can be connected with the fourth and the fifth (n-3 = 5-3 = 2 nodes). Finally, the fourth node can only be connected with the fifth node (n-4 = 5-4 = 1 node). This is the maximum number of activities that an AOA network with 5 nodes can have: 4+3+2+1 = 10 activities.

Figure 2. AOA fully connected 5 nodes network.



But, how do we generalize this result? Notice that for n nodes, the first node can have a maximum of n-1 connections or activities with all other nodes, the second node can have a total of n-2 activities, the third node n-3 activities and so on until the penultimate node that can have only one activity. Equation (2) indicates this sum.

$$\text{Max}(N) = (n-1) + (n-2) + (n-3) + \ldots + 1 \qquad (2)$$

In order to obtain the value of this sum, the previous list, but reversed in order is shown in equation (3).

$$\text{Max}(N) = 1 + 2 + 3 + \ldots + (n-1) \qquad (3)$$

Adding equations (2) and (3) yields equation (4), which is two times the value of Max(N).

$$2\,\text{Max}(N) = n + n + n + \ldots + n \qquad (4)$$

Equation (4) has number n added n-1 times, so that equation (5) must hold.

$$2\,\text{Max}(N) = n(n-1) \qquad (5)$$

From equation (5) we solve for Max(N) as indicated in equation (6).

$$\text{Max}(N) = n(n-1)/2 \qquad (6)$$

Table 1. Adjacency matrix for the fully connected activity network of 5 nodes in Figure 4.

Node Node	2	3	4	5
1	(1)→(2)	(1)→(3)	(1)→(4)	(1)→(5)
2		(2)→(3)	(2)→(4)	(2)→(5)
3			(3)→(4)	(3)→(5)
4				(4)→(5)

Thus, let N_{Min} be the minimum number of activities in the least complex of all possible activity networks as indicated by equation (1), N_{Max} be the maximum number of activities in a fully connected AOA network as indicated by equation (6), N the actual number of activities some specific activity network has and n the total number of its nodes, then the network complexity coefficient, *eta* (η), is given according to equation (7).

$$\eta = \frac{N - N_{Min}}{N_{Max} - N_{Min}} = \frac{N-(n-1)}{\frac{n(n-1)}{2} - (n-1)} = \frac{2(N-n+1)}{(n-1)(n-2)},\ n > 2 \qquad (7)$$

A value of η = 0 indicates minimal density for AOA network complexity, and a value η = 1 indicates maximum density for AOA network complexity. To illustrate the validity of equation (7), consider an AOA network of 5 nodes. If $N = N_{Min} = 5-1 = 4$, η = 0 when using equation (7) with N = 4 and n = 5, and if $N = N_{Max} = 10$, η = 1 when using N = 10 and n = 5 in equation (7). But, what about intermediate values? The minimum of activities for an AOA network of 5 nodes is 4 and the maximum is 10. Between 4 and 10 there are the numbers 5, 6, 7, 8 and 9. If we choose the middle value of N = 7, it should give us a density of 0.5, and it is precisely so, since when applying equation (7) with N = 7 and n = 5, we get η = 0.5.

ACTIVITY ON NODE (AON) NETWORK COMPLEXITY

The AOA network representation and the corresponding adjacency matrix make it easy to come up with the network complexity coefficient *eta* (η). Nevertheless, it is algorithmically difficult to implement the Critical Path Method (CPM) using AOA. Thus, AON network representation is required. However, AON requires a somewhat more elaborated thinking in order to come up with the network complexity coefficient *eta* (η).

Table 2. Fully connected AON adjacency matrix.

Activity	1	2	3	…	N-1	N	N+1
Activity							
0	*1*						
1		1	1	…	1	1	
2			1	…	1	1	
3					1	1	
⋮						⋮	
N-1						1	
N							*1*

The first and most important thing to consider is the limitation on the adjacency matrix imposed by the acyclicity requirement of a CPM activity network. In AOA representation the rows and columns denote precedence relationships while the ones inside the matrix denote activities. In the AON representation, the nodes themselves denote activities and the ones in the adjacency matrix denote precedence relationships. However, the same requirement in AON representation exists, that is, all rows/columns must be eliminated in order to prove that the network is acyclic. Thus, a fully connected AON adjacency matrix would be one with an upper triangular matrix full of ones, while the rest remain zeroes. Let N be the total number of activities in an AON network. In such case, 0 denotes the starting node and N+1 denotes the finishing node. The starting and finishing nodes are not activities, since they simply indicate the beginning and end of the AON network, respectively.

Schedule (www.schedulemanagement.net) does not allow precedence relationships between regular activities (numbered from 1 to N) and starting (numbered 0) or finishing (numbered N+1) nodes. Thus, the upper triangular matrix can only include activities from 1 to N. Also, it is necessary to indicate a beginning activity and an ending activity. For simplicity, in a fully connected AON network such activities are the first and last ones. Table 2 illustrate such fully connected AON adjacency matrix.

Let p denote the number of precedence relationships a given AON representation has as indicated in its corresponding adjacency matrix. The precedence relationships denoted by a one in bold typeset can be established by the user, whereas the precedence relationships denoted by a gray bold one in italics typeset are the default ones created by the system in order to link the starting node and the finishing nodes. Table 2 indicates the maximum number of precedence relationships or Max(p). In this case, it is easy to see that Max(p) is given by equation (8).

$$\text{Max}(p) = N(N-1)/2 + 2 \qquad (8)$$

A minimally connected AON network would be one such that starting node 0 is connected to activity 1, activity 1 is connected to activity 2, activity 2 is connected to activity 3, and so on, until activity N is connected to finishing node N+1. Table 3 illustrates the adjacency matrix for this minimally connected AON activity network.

Table 3. Minimally connected AON adjacency matrix.

Activity / Activity	1	2	3	…	N-1	N	N+1
0	*1*						
1		1					
2			1				
3				⋱			
⋮					⋱		
N-1						1	
N							*1*

A minimally connected AON network would be one such that starting node 0 is connected to activity 1, activity 1 is connected to activity 2, activity 2 is connected to activity 3, and so on, until activity N is connected to finishing node N+1. Table 3 illustrates the adjacency matrix for this minimally connected AON activity network.

Once again, it is easy to see that the number of precedence relationships for a minimally connected adjacency matrix is given by equation (9).

$$\text{Min}(p) = N+1 \qquad (9)$$

Let p_{Max} be the maximum number of precedence relationships as indicated by equation (8) and p_{Min} be the minimum number of precedence relationships as indicated by equation (9), then the network complexity coefficient, *eta* (η), is given by equation (10).

$$\eta = \frac{p - p_{Min}}{p_{Max} - p_{Min}} = \frac{p-(N+1)}{\frac{N(N-1)}{2}+2-(N+1)} = \frac{2(p-N-1)}{N(N-1)-2(N+1)+4}, \; N > 3 \qquad (10)$$

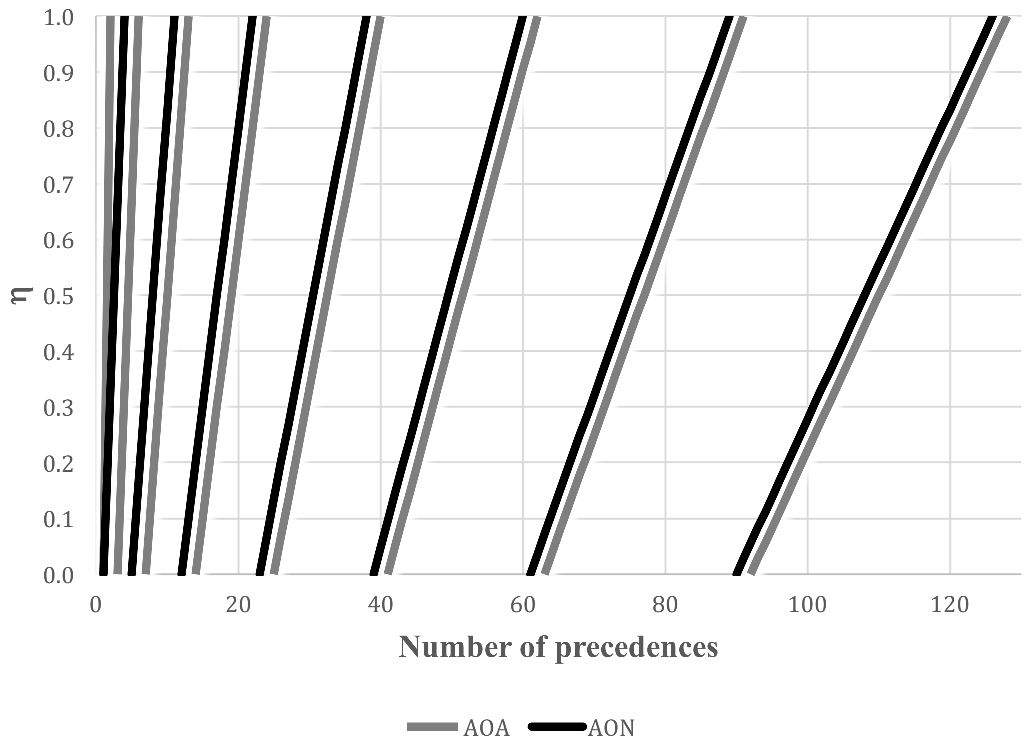
The restriction that N > 3 in equation (10) comes from realizing that the denominator in equation (10) has to be different than zero. In an activity network with no precedence relationships, *Schedule* would create N+N links so that all activities are connected to the starting node and the finishing node. In the case of activities having no precedence relationships, *Schedule* would create a precedence relationship with the starting node (node 0), whereas those activities that are no predecessors to any activity would have a link to the finishing node (node N+1).

RESULTS

There can be a certain number of nodes. In AOA such nodes denote precedence relationships, while the arrows indicate activities. In AON the nodes indicate activities and the arrows precedence relationships. However, regardless of the notation selected, the network and its nodes must be acyclic.

Then, given a certain number of nodes (going from 3 for AOA and 4 for AON), all the possible number of precedence relationships is considered and the complexity network (η) is calculated for each case. Figure 3 shows the results. Notice that the novelty of this approach is that it only considers network density, not network structure. As such, it is able to produce a very intuitive complexity coefficient called *eta* (η) that measures how close to the maximum network density any given network is, where 0 ≤ η ≤ 1.

Figure 3. Complexity network calculation with respect to the total number of activities as the number of activities to be considered increases.



DISCUSSION AND CONCLUSION

Thanks to the fact that an activity network for project management needs to be acyclic, it is possible to set a minimum and a maximum bound for the number of precedence relationships that exist within the network. This allows to define a complexity coefficient that focuses entirely on network density. It needs not consider network structure because the network is restricted to be acyclic. Thus, the complexity coefficient defined is able to focus entirely on network density, that is, how close to the maximum number of precedence relationships the network is, and, as such, is able to define the complexity coefficient as a value between zero and one.

The requirement for η in the case of AOA is that n > 2 and for AON that N > 3. The gray line indicates an AOA network, whereas the black line indicates an AON network. The first line (in gray for an AOA network) is for the case in which n > 2. In that case, the maximum possible number of precedence relationships (activities), indicated by equation (6), is 3, so that N = 2 and 3. Then follows a black line for an AON network with 4 activities (N = 4), with a minimum number of activities of 5 and a maximum number of activities of 8. Thus, p = 5, 6, 7, and 8. In each case, η is calculated. The next line (gray) is for an AOA network with a total of N = 4 activities. In that case the minimum number of activities goes from 4 to 6, and the complexity coefficient (η) is calculated for each case. All the cases for a total of 3 to 10 activities in the case of an AOA network and for a total of 4 to 10 activities in the case of an AON network are illustrated in Figure 3. As it can be seen, the relationship between the number of activities and the complexity coefficient is linear, although the slope changes depending on how many nodes are considered. In the case of an AOA representation, the nodes indicate precedence relationships, and according to equations (1) and (6) the minimum and maximum number of activities to be in the network is calculated. For AON representation, the nodes indicate activities, and the maximum and minimum possible number of activities, given by equations (8) and (9), respectively, is considered and the complexity coefficient calculated in each case.

One straight line indicates how the complexity coefficient (η) changes as its connections to other activities within a given activity network structure increases.

The conclusion is that the complexity coefficient (η) gives a very intuitive idea of how close to the maximum number of possible connections in the activity network each network being considered is.

References

Dehmer, M., Barbarini, N., Varmuza, K., & Graber, A. (2009). A Large Scale Analysis of Information-Theoretic Network Complexity Measures Using Chemical Structures. *PLOS One*, *4*(12), 1-13.

Elsayed, E. A. & Boucher, T. O. (1994). *Analysis and control of production systems*. Prentice Hall.

Emmert-Streib, F. & Dehmer, M. (2012). Exploring Statistical and Population Aspects of Network Complexity. *PLOS One*, *7*(5), 1-17.

Kaimann, R. (1974). Coefficient of Network Complexity. *Management Science*, *21*(2), 172-177.

Keller, C., Berger, F., Liepelt, S., & Lipowsky, R. (2013). Network Complexity and Parametric Simplicity for Cargo Transport by Two Molecular Motors. *Journal of Statistical Physicsk, 150*, 205-234.

Rigterink, D. & Singer, D. J. (2014). A network complexity metric based on planarity and community structure. *Journal of Complex Networks*, *2*, 299-312.

Zenil, H., Kiani, N. A., & Tegnér, J. (2018). A Review of Graph and Network Complexity from an Algorithmic Information Perspective. *Entropy*, *20*, 1-15.

Acknowledgements

The author wishes to thank my Ph.D. thesis supervisor, Dr. Normal P. Archer, while doing my Ph.D. at the Michael G. DeGroote School of Business at McMaster University, and the financial support received as teaching and research assistantships.