ISSN: 2634-9221
European Journal of Applied Sciences; Vol. 14 No. 02 (2026) (577-581)
https://doi.org/10.14738/aivp.1402.20241

Review of Entanglement Entropy in Non-Hermitian Quantum Many-Body Systems: From Area Laws to Exceptional Criticality

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Abstract: Entanglement entropy serves as a fundamental diagnostic for quantum many-body correlations, distinguishing quantum entanglement from classical stochastic fluctuations. This review systemati- cally explores the scaling behaviors of entanglement entropy, beginning with the foundational area law in Hermitian systems and progressing to the exotic regimes of non-Hermitian physics. We focus on the mathematical apparatus of the Fisher-Hartwig theorem in fermionic chains, the biorthogonal formalism in non-unitary systems, and the emergence of negative entanglement entropy driven by exceptional boundary states and non-unitary conformal field theories.

INTRODUCTION

In the paradigm of quantum pure states, the entropy of a subsystem arises not from classical ignorance but from the intrinsic non-locality of quantum entanglement. For quantum many-body systems characterized by local interactions, entanglement is typically localized at the boundary interface. This physical intuition is formalized by the”area law” which states that the groundstate entanglement entropy (EE) of a subsystem scales with the area of its boundary rather than its volume[1,2]. This phenomenon is profoundly interdisciplinary, bridging the gap between condensed matter physics and black hole thermodynamics, where it mirrors the Bekenstein-Hawking entropy. While gapped systems generally satisfy the area law, critical systems with vanishing gaps exhibit lo-garithmic corrections. Such scaling behavior provides a direct window into the central charge of the underlying conformal field theory (CFT) and the decay characteristics of correlation functions[1].

The exploration of entanglement has recently extended into the realm of non-Hermitian physics, a field that has transitioned from a phenomenological description of open systems to a fundamental framework for non-unitary dynamics. The discovery of PT-symmetric Hamiltonians demonstrated that non-Hermitian systems could possess entirely real spectra, while the identification of exceptional points (EPs)-singularities where eigenvalues and eigenvectors coalesce-umveiled new topological phases[3]. Experimental progress in platforms such as ultracold atoms with controlled loss and nonreciprocal photonic lattices has further highlighted the non-Hermitian skin effect (NHSE), where an extensive number of bulk modes collapse onto the boundaries. This breakdown of the conventional bulk-boundary correspo-ndence necessitates a generalized Brillouin zone[4] approach and fundamentally reconfigures the entanglement landscape, as the localization of wavefunctions directly influences the distribution of quantum information.

As non-Hermitian systems lack the unitary constraints of standard quantum mechanics, the traditional definition of von Neumann entropy must be re-evaluated. The emergence of non-Hermitian entanglement entropy is rooted in the necessity of capturing the unique criticality of non-unitary systems, such as those described by bc-ghost CFTs[5-8]. Central to this development is the biorthogonal basis framework, which utilizes both right and left eigenvectors to define a physically consistent reduced density matrix. This evolution has led to the discovery of remarkable phenomena, including negative entanglement entropy and the doubling of central charges at phase transitions. By examining the interplay between non-reciprocity, dissipation, and quantum correlations, the study of non-Hermitian EE provides a robust diagnostic for topological phases and the localizationdelocali-zation transitions in non-unitary quantum matter.

BOSONIC HARMONIC CHAINS AND THE SCALING LIMITS

For a 1D nearest-neighbor bosonic harmonic chain[9], the Hamiltonian is represented by $H = \frac{1}{2}\sum_{i,j}(p_i P_{ij} p_j + x_i X_{ij} x_j)$,where X and $P$are real symmetric positivedefinite matrices. In a translation-invariant model X is a circulant matrix whose first row elements are $(a,b,0,\ldots,0,b)$,where a denotes the on-site potential and b represents the nearest-neighbor coupling. To ensure a stable gapped system, the condition $a>2/b/$ must be satisfied. Physically, the operator norm $||\mathbf{X}||$ determines the sound velocity $v \sim ||X||^{1/2}$. In the gapped regime[10], the entanglement entropy $S(\rho_l)$ is bounded by a constant independent of the subsystem size N :

$$S(\rho_I)\leq\frac{1}{4}\log_2\left(\frac{a+2/b/}{a-2/b/}\right)=\frac{1}{2}\log_2\left(\frac{\|\mathbf{X}\|^{1/2}}{\Delta E}\right) \tag{1}$$

where $\Delta E=\sqrt{a-2|b|}$ is the spectral gap. As the system approaches the critical point $(a\rightarrow 2|b|)$, the correlatiom length diverges, and the system is described by a massless Klein-Gordon field. In this limit[11], the entanglement diverges logarithmically as $E_N(\rho,l)=\frac{1}{4}\log_2(1+4N^2/m^2)$ which asymp-totically approaches $\frac{1}{2}\log N$. This contrast confirms that while the gap protects the area law, criticality inherently drives a logarithmic violation proportional to the effective degrees of freedom.

FISHER-HARTWIG THEOREM AND FERMIONIC CRITICALITY

In fermionic systems like the XY model, the logarithmic scaling of EE is intrinsically linked to the topology of the Fermi surface. For a 1D free-fermion system, the ground-state entropy is determined by the spectrum of the correlation matrix V_n, which possesses a Toeplita structure. A rigorous lower[12] bound for the entropy is given by $S(\rho_l)\geq-\frac{1}{2}\log_2/\det(V_n)/$. The critical Ising model corresponds to a jump discontinuity in the symbol function $g(\varphi)$ at $\varphi=0$. According to the Fisher-Hartwig theorem, the determinant of such a Toeplitz matrix scales as $\det(V_n)\sim Cn^{-1/4}$ for $n\rightarrow\infty$. Substituting this into the entropy bound yields[13]:

$$S(\rho_I)\geq-\frac{1}{2}\log_2(Cn^{-1/4})=\frac{1}{8}\log_2 n+O(1) \tag{2}$$

This confirms that $S(\rho_l)=\Omega(\log_2 n)$, establishing that the logarithmic divergence is a direct consequence of the Fermi surface “jump”[14]. Interestingly, even in gapped systems, long-range interactions (e.g., 1/r powerlaw hopping) can sustain multiple Fermi surface jumps $(k > 0)$ [15]. In such cases, the Fisher-Hartwig theorem predicts $\det(V_n) \sim Cn^{-k/4}$, leading to $S(\rho_I) \geq \frac{k}{8} \log_2 n$.This implies that the area law is not guaranteed by a gap alone; rather, the connectivity of the interactions and the resulting Fermi surface topology are the ultimate determinants of the scaling behavior.

LIEB-ROBINSON BOUND AND THE LOCALITY OF ENTANGLEMENT

The formal justification for the area law in 1D gapped local systems rests on the Lieb-Robinson bound, which constrains the velocity of information propagation[16] . For a Hamiltonian $H = \sum_j H_{j,j+1}$, the time evolution of a local operator A satisfes $\parallel [A(t), B] \parallel \leq C \parallel A \parallel \parallel B \parallel e^{-\mu(d - v|t|)}$, where v is the Lieb-Robinson velocity. This fnite velocity implies that the correlation between a subsystem I and its environment decays exponentially beyond a certain range. By constructing an approximately local mapping, one can show that the reduced density matrix ρ_I is exponentially close to a state $\widetilde{\rho}_I$ supported on a boundary region of fixed dimension[17]: $\parallel \rho_I - \tilde{\rho}_I \parallel_1 \leq e^{-cn}$.Consequenly, the entropy $S(\rho_I)$ is bounded by a constant proportional to the boundary degrees of freedom, $S(\rho_I) \leq \mathrm{const}$. This theoretical framework reinforces that in Hermitian systems with short-range interactions, the entanglement is a purely boundary phenomenon.

NON-HERMITIAN FORMALISM AND BIORTHOGONAL ENTANGLEMENT

In non-Hermitian systems, the many-body ground state is defined through biorthogonal eigenvectors[18,19]: $|G_R\rangle = \prod_{a \in \mathrm{occ}} \widehat{\psi}_a^{\dagger} |0\rangle$ and $\langle G_L| = \prod_{a \in \mathrm{occ}} \widehat{\psi}_{L,a} |0\rangle$, satisfying $H|G_R\rangle = E|G_R\rangle$ and $H^{\dagger}\langle G_L| = E^{*}\langle G_L|$. The reduced density matrix $\rho_A = \mathrm{Tr}_B |G_R\rangle\langle G_L|$ is non-Hermitian, necessitating modified entropy definitions. One approach employs a model-dependent trace modification[20,21], such as $\rho_A = \mathrm{Tr}_B(\rho^{2\sigma^z}\rho)$ in non-Hermitian XXZ chains, which yields $S = \ln q + q^{-1}$. A more universal, model-independent approach utilizes the modulus of the density matrix:

$$S = -\mathrm{Tr}(\rho_A \ln|\rho_A|), \quad S^{(n)} = \frac{1}{1-n} \ln \mathrm{Tr}\left(\rho_A |\rho_A|^{n-1}\right) \tag{3}$$

This modulus-based definition consistently identifies negative central charges (c=-2) in non-unitary critical systems[22-24], such as the non-Hermitian SSH model[25]. In these systems, the logarithmic scaling $S_A \sim -\frac{2}{3} \ln \sin\left(\frac{\pi L_A}{L}\right)$ reflects the bc-ghost CFT structure[6-8].Furthermore, exceptional boundary (EB) states in topological non-Hermitian models lead to anomalous occupation probabilities exceeding unity, resulting in negative real parts of the entanglement entropy scaling as $\mathrm{Re}S \sim -\frac{1}{3} \ln L$[26].This highlights that non-Hermiticity can induce correlations that are fundamentally different from their Hermitian counterparts, particularly near exceptional points.

CONCLUSION

The evolution of entanglement entropy from Hermitian area laws to non-Hermitian logarithmic scaling reveals the deep connection between interaction locality, spectral gaps, and the underlying feld theory. While the LiebRobinson bound protects the area law in gapped local systems, long-range interactions or non-Hermiticity can bypass these constraints, leading to fractal Fermi surfaces or negative central charges. As experimental techniques continue to advance in simulating non-unitary dynamics. entanglement entropy will remain an indispensable tool for characterizing the topology and information content of open quantum matter.

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