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Idempotent Cayley Graph of the Ring (Z_n,⊕,⊙)

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ABSTRACT
The idempotent Cayley graph of the ring (Z_n,⊕,⊙) is the Cayley graph associated with the symmetric set consisting of idempotent elements and their inverses in the group (Z_n,⊕). In this paper, we derive some basic properties of the idempotent elements and construct the idempotent Cayley graph. It is shown that this graph is connected and Hamiltonian. Further, if $n = 2^\alpha, \alpha > 1$, this graph is bipartite.
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Keywords: Idempotent element, idempotent Cayley graph, connected, bipartite, Hamiltonian.

INTRODUCTION
The study of graphs associated with algebraic structures, particularly groups and rings, is an active field of research in recent years. Many graph theorists, to mention some [1,2,4,5,8,13,14,15] have studied various aspects of graphs arising from rings and groups. Madhavi et.al [7,11,12] studied Cayley graphs associated with arithmetic functions, like Euler-totient function, quadratic residues modulo a prime p, divisor function from Number theory [3]; zero-divisors and nilpotent elements in the ring (Z_n,⊕,⊙). Somayyeh Razaghi and Shervin Sahebi [14] have studied some kind of topological indices of the graph associated with the idempotent elements of the ring (Z_n,⊕,⊙). This graph has Z_n as the vertex set V and the edge set $E = \{(\bar{a}, \bar{b}) \,/\, \bar{a}, \bar{b} \in Z_n \;\; and \;\; \bar{a} + \bar{b} \; is \; an \; idempotent \; element\}$. In our study, we introduce another class of graphs, namely, the Cayley graph associated with the symmetric subset arising out of the idempotent elements of the ring (Z_n,⊕,⊙).

The degree $d(v)$ of a vertex v in a graph G is the number of edges incident with each vertex $v \in G$. If degree of each vertex in G is same, say r, then G is called r −regular graph. A graph is a complete graph, if every vertex is adjacent to all other vertices of the graph. A walk in a graph G is an alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, \dots, e_n, v_n$, beginning and ending with vertices, in which each edge is incident with the two vertices immediately preceding and following it. A walk is closed if $v_0 = v_n$. A closed walk in which all the edges are distinct is called a circuit. An Eulerian circuit in a graph G is a circuit containing every edge of G and G is an Eulerian graph if it contains an Eulerian circuit.

A cycle in a graph is a sequence of distinct vertices $v_1, v_2, \dots, v_{r-1}, v_r$ such that $(v_1, v_2), (v_2, v_3), \dots, (v_{r-1}, v_r), (v_r, v_1)$ are edges. It is denoted by $(v_1, v_2, \dots, v_r, v_1)$ and r is called its length. A Hamilton cycle in a graph G is a cycle containing every vertex of G and G is called a Hamiltonian graph if it contains a Hamilton cycle. A bipartite graph is a graph, whose vertex set can be partitioned into two disjoint subsets X and Y (that is, $V(G) = X \cup Y, X \cap Y = \emptyset$) such that each edge has one end in X and other end in Y.

For standard terminology and notions in graph theory, we refer Bondy and Murty [6] and Harary [9] and for algebra Gallian [10].

IDEMPOTENT ELEMENTS IN THE RING $(Z_n,\oplus,\odot)$

In this section, some properties of idempotent elements in the ring $(Z_n,\oplus,\odot)$ of residue classes modulo n, $n \geq 1$ an integer, that are needed for defining the Idempotent Cayley Graph associated with the group $(Z_n,\oplus)$ are obtained.

Definition 2.1: Let $(X,\cdot)$ be a group. A subset S of X is called asymmetric subset of X if $s^{-1} \in S, \forall\, s \in S$.

Definition 2.2: Let $(R,+,\cdot)$ be a ring. An element a in R is called an **idempotent element** if $a^2 = a$.

Lemma 2.3: Let I denote the set of idempotent elements of the ring $(R,+,\cdot)$. Then I is not a symmetric subset of the group $(R,+)$.

Proof: Let $a \in I$, and let $-a$ be its inverse in the group $(R,+)$. By the elementary properties of rings $a \in I$ implies that $a^2 = a$.

$\Rightarrow (-a)^{\mathbf{2}} = (-a)(-a) = -[(a)(-a)] = -[-(a\,a)] = a\,a = a^2 = a.$

That is, $(-a)^2 \neq -a$, so that $-a \notin I$.

This shows that I is **not** a symmetric subset of the group $(R,+)$.

The following lemma can be easily proved by using the properties of ring isomorphism.

Lemma 2.4: Let $(R,+,\cdot)$ and $(S,+,\cdot)$ be rings and let $\psi: R \longrightarrow S$ be an isomorphism. Then the image of an idempotent element in $(R,+,\cdot)$ under ψ is an idempotent element in $(S,+,\cdot)$. Further, the pre-image of an element in $(S,+,\cdot)$ under ψ is an idempotent element in $(R,+,\cdot)$.

Idempotent Elements in the Ring $(Z_n,\oplus,\odot)$ of Integer Modulo n, $n \geq 1$ an Integer

Let $n \geq 1$ be an integer. In the ring $(Z_n,\oplus,\odot)$ of integers modulo n, an element $\bar{a} \in Z_n$ is called an idempotent element if $(\bar{a})^2 = \bar{a}$, or, $\bar{a} \odot \bar{a} = \bar{a}$.

In the study of the idempotent Cayley graph of the ring (Z_n,⊕,⊙), the idempotent elements and their number play a crucial role. A formula for determining the number of idempotent elements is obtained in the ring (Z_n,⊕,⊙) in the following theorem.

Theorem 2.5: Let $n \geq 1$ be an integer. Then the number of idempotent elements in the ring (Z_n,⊕,⊙) is 2^k, where k is the number of distinct prime divisors of n.

Proof: Let $n \geq 1$ be an integer and let $n = {p_1}^{\alpha_1} {p_2}^{\alpha_2} \ldots {p_k}^{\alpha_k}$, where $p_1 < p_2 < \cdots < p_k$ are distinct primes and $\alpha_i \geq 1, 1 \leq i \leq k$ are integers.

By the Chinese Remainder Theorem,

$$Z_n \cong Z_{{p_1}^{\alpha_1}} \times Z_{{p_2}^{\alpha_2}} \times \ldots \times Z_{{p_k}^{\alpha_k}}$$

under the isomorphism $\psi: Z_m \longrightarrow Z_{{p_1}^{\alpha_1}} \times Z_{{p_2}^{\alpha_2}} \times \ldots \times Z_{{p_k}^{\alpha_k}}$ given by

$$\psi(\bar{a}) = \left(\bar{a} + Z_{{p_1}^{\alpha_1}}, \bar{a} + Z_{{p_2}^{\alpha_2}}, \ldots, \bar{a} + Z_{{p_k}^{\alpha_k}}\right). \_\_\_\_\_\_\_\_\_\_ (1)$$

First we observe that, for any prime number p and $r \geq 1$ an integer, the only idempotent elements in the ring $\left(Z_{p^r}, ⊕, ⊙\right)$ are $\bar{0}$ and $\bar{1}$.

For $r \geq 1$, we shall show that 0 and 1 are the only solutions of the congruence equation

$$x^2 \equiv x \ (mod\ p^r). \_\_\_\_\_\_\_\_\_\_ (2)$$

Clearly $x = 0$ is a solution of (2).

Suppose $1 \leq x \leq p^r - 1$ is another solution of (2). Then

$$x^2 \equiv x \ (mod\ p^2).$$

Let $x = sp^t$, where $(s, p) = 1$ and $0 \leq 1 < r$. Then

$$x^2 \equiv x \ (mod\ p^r) \Rightarrow (sp^t)^2 \equiv sp^t \ (mod\ p^r)$$

$$\Rightarrow (s^2 p^{2t} - sp^t) \equiv 0 \ (mod\ p^r)$$

(dividing the congruence equation by p^t, since p is a prime)

$$\Rightarrow s(sp^t - 1) \equiv 0 \ (mod\ p^{r-t})$$

$$\Rightarrow (sp^t - 1) \equiv 0 \ (mod\ p^{r-t}), \text{ since } (s, p) = 1$$

$$\Rightarrow (sp^t, p^{r-1}) = 1. \_\_\_\_\_\_\_\_\_\_ (3)$$

Again since $(s, p) = 1$, we have $(sp^t, p^{r-t}) = p$ and this is a contradiction to the fact that $(sp^t, p^{r-1}) = 1$ (by (3)), unless $t = 0$.

Then $t = 0$ gives $x = sp^0 = s \equiv 1\ (mod\ p)$, since $(s, p) = 1$.

This shows that $x = 1$ is the only solution of $x^2 \equiv x\ (mod\ p^r)$, other than $x = 0$.
Thus 0 and 1 are the only two solutions of the congruence equation $x^2 \equiv x\ (mod\ p^r)$.

From this it follows that $\bar{0}$ and $\bar{1}$ are the only two idempotent elements in $(Z_{p^r}, \oplus, \odot)$. So $(e_1, e_2, .., e_k) \in Z_{p_1{}^{\alpha_1}} \times Z_{p_2{}^{\alpha_2}} \times \ldots \times Z_{p_k{}^{\alpha_k}}$, where $e_i = \bar{0}$, or, $\bar{1}$, $1 \le i \le k$ are the only idempotent elements in $Z_{p_1{}^{\alpha_1}} \times Z_{p_2{}^{\alpha_2}} \times \ldots \times Z_{p_k{}^{\alpha_k}}$ and their number is 2^k, where k is the number of distinct prime divisors of n.

By the Lemma 2.4, the idempotent elements of the ring Z_n are thus simply the pre-images of these idempotent elements in $Z_{p_1{}^{\alpha_1}} \times Z_{p_2{}^{\alpha_2}} \times \ldots \times Z_{p_k{}^{\alpha_k}}$ under the isomorphism ψ. So the pre-image of an idempotent element in $\prod_{i=1}^{k} Z_{p_i{}^{\alpha_i}}$ is also an idempotent element in Z_n.

This shows that the number of idempotent elements in the ring $(Z_n, \oplus, \odot)$ is 2^k, where k is the number of distinct prime divisors of n.

Example 2.6: Consider $(Z_{12}, \oplus, \odot)$. Here $12 = 2^2.3$, so that 2 and 3 are the only two distinct prime divisors of 12 and thus the number of idempotent elements in $(Z_{12}, \oplus, \odot)$ is $2^2 = 4$.

Also $Z_{12} \cong Z_{2^2} \times Z_3 = Z_4 \times Z_3$.

The idempotent elements in $(Z_4, \oplus, \odot)$ are $\bar{0}$ and $\bar{1}$ and similarly the idempotent elements in $(Z_3, \oplus, \odot)$ are also $\bar{0}$ and $\bar{1}$. So the idempotent elements in $Z_4 \times Z_3$ are $\{(\bar{0}, \bar{0}), (\bar{0}, \bar{1}), (\bar{1}, 0), (\bar{1}, \bar{1})\}$.

Let $d = (\bar{0}, \bar{1}) \in Z_4 \times Z_3$. Then $d = (\bar{0}, \bar{1}) = (\bar{0} + 4Z, \bar{1} + 3Z)$.

If $\bar{a} \in Z_{12}$ is the pre-image of d under the isomorphism $\psi: Z_{12} \to Z_4 \times Z_3$, then $\psi(\bar{a}) = (\bar{a} + Z_4, \bar{a} + Z_3) = (\bar{a} + 4Z, \bar{a} + 3Z)$, so that $(\bar{a} + 4Z, \bar{a} + 3Z) = d = (\bar{0}, \bar{1}) =$
$(\bar{0} + 4Z, \bar{1} + 3Z)$ and this shows that $a \equiv 0\ (mod\ 4)$ and $a \equiv 1\ (mod\ 3)$.

Now $4Z = \{0, \mathbf{4}, 8, 12\}$. (Since we are considering Z_{12}, we need not go beyond 12) and $3Z + 1 = \{1, \mathbf{4}, 7, 11\}$.

Since 4 is the common element to $4Z$ and $3Z + 1$, $\bar{4} \in Z_{12}$ is the pre-image of $(\bar{0}, \bar{1})$.

Observe that $(\bar{4})^2 = \bar{4} \odot \bar{4} = \overline{16} = \bar{4}\ (mod\ 12)$ and thus $\bar{4}$ is an idempotent element in $(Z_{12}, \oplus, \odot)$.

In this way, the pre-images of $(\bar{0}, \bar{0})$, $(\bar{1}, \bar{0})$ and $(\bar{1}, \bar{1})$ can be determined by using the above process and these are given in the Table 1.

Table 1: Table for pre-images

$\mathbf{0}(\boldsymbol{mod}\ \mathbf{4})$	$\mathbf{1}(\boldsymbol{mod}\ \mathbf{4})$	$\mathbf{0}(\boldsymbol{mod}\ \mathbf{3})$	$\mathbf{1}(\boldsymbol{mod}\ \mathbf{3})$
0	1	0	1
4	5	3	4
8	9	9	10
12	13	12	13

Pre-image of $(\bar{0},\bar{0})$ is $\bar{0}$, as the common element to $0(mod\ 4)$ and $0(mod 3)$ is $\bar{0}$.
Pre-image of $(\bar{0},\bar{1})$ is $\bar{4}$, as the common element to $0(mod\ 4)$ and $1(mod 3)$ is $\bar{4}$.
Pre-image of $(\bar{1},\bar{0})$ is $\bar{9}$, as the common element to $1(mod\ 4)$ and $0(mod 3)$ is $\bar{9}$.
Pre-image of $(\bar{1},\bar{1})$ is $\bar{1}$, as the common element to $1(mod\ 4)$ and $1(mod 3)$ is $\bar{1}$.

So the idempotents in $(Z_{12},\oplus,\odot)$ are $\bar{0},\bar{1},\bar{4},\bar{9}$. Observe that $\bar{0}^2=\bar{0}$, $\bar{1}^2=\bar{1}$, $\bar{4}^2=\overline{16}=\bar{4}$, $\bar{9}^2=\overline{81}=\bar{9}$.

Remark: The method of finding the idempotent elements and their number as given in Theorem 2.5 is applicable for any integer $n\geq 1$. However for manageable smaller numbers n, the idempotent elements $\bar{a}\in Z_n$ can be calculated by checking whether $(\bar{a})^2=\bar{a}$. For example:

1. For $n=6$, we have $(\bar{0})^2=\bar{0}$, $(\bar{1})^2=\bar{1}$, $(\bar{2})^2=\bar{4}$, $(\bar{3})^2=\bar{9}=\bar{3}$, $(\bar{4})^2=\overline{16}=\bar{4}$ and $(\bar{5})^2=\overline{25}=\bar{1}$, so that $\bar{0},\bar{1},\bar{3},\bar{4}$ are the idempotent elements in the ring $(Z_6,\oplus,\odot)$. Here $6=2\times 3$, product of two distinct primes, so that the number of idempotent elements in the ring $(Z_6,\oplus,\odot)$ is $2^2=4$
2. For $n=7$, we have $(\bar{0})^2=\bar{0}$, $(\bar{1})^2=\bar{1}$, $(\bar{2})^2=\bar{4}$, $(\bar{3})^2=\bar{9}=\bar{2}$, $(\bar{4})^2=\overline{16}=\bar{2}$, $(\bar{5})^2=\overline{25}=\bar{4}$ and $(\bar{6})^2=\overline{36}=\bar{1}$ so that $\bar{0},\bar{1}$ are the only idempotent elements in the ring $(Z_7,\oplus,\odot)$. Here $n=7$ is a single prime, so that the number of idempotent elements in the ring $(Z_6,\oplus,\odot)$ is $2^1=2$.
3. For $n=8=2^3$, which is a power of a single prime. We have $(\bar{0})^2=\bar{0}$, $(\bar{1})^2=\bar{1}$, $(\bar{2})^2=\bar{4}$, $(\bar{3})^2=\bar{9}=\bar{1}$, $(\bar{4})^2=\overline{16}=\bar{0}$, $(\bar{5})^2=\overline{25}=\bar{1}$, $(\bar{6})^2=\overline{36}=\bar{4}$ and $(\bar{7})^2=\overline{49}=\bar{1}$ so that $\bar{0},\bar{1}$ are the only idempotent elements in the ring $(Z_7,\oplus,\odot)$ and whose number is $2^1=2$.

THE IDEMPOTENT GRAPH OF THE RESIDUE CLASS RING $(\boldsymbol{Z_n},\oplus,\odot)$

Let $n\geq 1$ be an integer. Let I be the set of idempotent elements in the ring $(Z_n,\oplus,\odot)$. Then by the Lemma 2.2, I is **not** a symmetric subset of the group $(Z_n,\oplus)$. Let $I^*=I-\{\bar{0}\}$ and let $(I^*)^{-1}=\{\overline{n-a}/\bar{a}\in I^*\}$, be the set of inverse elements of I^* in the group $(Z_n,\oplus)$. Then clearly $\mathbb{I}=I^*\cup(I^*)^{-1}$ is a symmetric subset of $(Z_n,\oplus)$. Since $\bar{1}\in I$, we have $\overline{n-1}\in(I^*)^{-1}$ and thus $\bar{1},\overline{n-1}\in\mathbb{I}$.

Definition 3.1: Let $n\geq 1$ be an integer. The **idempotent Cayley graph** of the ring $(Z_n,\oplus,\odot)$ is the graph whose vertex set V is Z_n and the edge set $E\{(\bar{a},\bar{b})/\bar{a},\bar{b}\in\mathbb{I}\ and\ either\ \bar{a}-\bar{b}\in\mathbb{I}, or, \bar{b}-\bar{a}\in\mathbb{I}\}$. This graph is denoted by $\Gamma(Z_n)$.

Clearly $\Gamma(Z_n)$ is a simple graph (having no loops and parallel edges).

Remark: If $\bar{0}$ is **not** excluded, from I to set I^*, then $\bar{0} \in \mathbb{I}$ and for every $\bar{a} \in Z_n$, $\bar{a} - \bar{a} \in \mathbb{I}$ and for all $\bar{a} \in Z_n$, $(\bar{a}, \bar{a})$ is a loop in the graph $\Gamma(Z_n)$. Thus $\Gamma(Z_n)$ contains a loop at every vertex. As we are seeking a loopless graph, $\bar{0}$ is deleted from $\mathbb{I}$.

Example 3.2: The idempotent Cayley graph $\Gamma(Z_6)$ of the ring $(Z_6, \oplus, \odot)$ is as follows. Here $I = \{\bar{0}, \bar{1}, \bar{3}, \bar{4}\}$, since $\bar{0}^2 = \bar{0}, \bar{1}^2 = \bar{1}, \bar{3}^2 = \bar{9} = \bar{3}, \bar{4}^2 = \overline{16} = \bar{4}$, so that $I^* = \{\bar{1}, \bar{3}, \bar{4}\}$ and $(I^*)^{-1} = \{\overline{6-1}, \overline{6-3}, \overline{6-4}\} = \{\bar{5}, \bar{3}, \bar{2}\}$. So $\mathbb{I} = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ is a symmetric set of the group $(Z_6, \oplus)$.

Table 2: Table for $\bar{a} - \bar{b}$ in $(Z_6, \oplus)$

$\bar{a} - \bar{b}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$
$\bar{0}$	$\bar{0}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$
$\bar{1}$	$\bar{1}$	$\bar{0}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$
$\bar{2}$	$\bar{2}$	$\bar{1}$	$\bar{0}$	$\bar{5}$	$\bar{4}$	$\bar{3}$
$\bar{3}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	$\bar{0}$	$\bar{5}$	$\bar{4}$
$\bar{4}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	$\bar{0}$	$\bar{5}$
$\bar{5}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	$\bar{0}$

There is an edge between the elements $\bar{a}$ and $\bar{b}$ for which $\bar{a} - \bar{b}$ (the shaded elements) belong to $\mathbb{I}$ in the Table 2. From the Table 2, it is evident that there is an edge between every pair of vertices in the vertex set $V = Z_6$ of $\Gamma(Z_6)$ and thus the graph $\Gamma(Z_6)$ is a complete graph as shown in Fig.1.

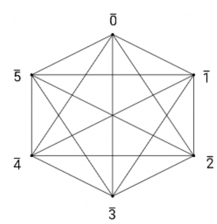

Fig.1: The graph $\Gamma(Z_6)$

Example 3.3: The idempotent Cayley graph $\Gamma(Z_{12})$ of the ring $(Z_{12}, \oplus, \odot)$ is as follows. Here $I = \{\bar{0}, \bar{1}, \bar{4}, \bar{9}\}$, so that $I^* = \{\bar{1}, \bar{4}, \bar{9}\}$, $(I^*)^{-1} = \{\overline{12-1}, \overline{12-4}, \overline{12-9}\} = \{\overline{11}, \bar{8}, \bar{4}\}$ and $\mathbb{I} = \{\bar{1}, \bar{4}, \bar{8}, \overline{11}, \bar{8}, \bar{3}\} = \{\bar{1}, \bar{3}, \bar{4}, \bar{8}, \bar{9}, \overline{11}\}$.

Table 3: Table for $\bar{a} - \bar{b}$ in $(Z_{12}, \oplus)$

$\bar{a} - \bar{b}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{7}$	$\bar{8}$	$\bar{9}$	$\overline{10}$	$\overline{11}$
$\bar{0}$	$\bar{0}$	$\overline{11}$	$\overline{10}$	$\bar{9}$	$\bar{8}$	$\bar{7}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$
$\bar{1}$	$\bar{1}$	$\bar{0}$	$\overline{11}$	$\overline{10}$	$\bar{9}$	$\bar{8}$	$\bar{7}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$
$\bar{2}$	$\bar{2}$	$\bar{1}$	$\bar{0}$	$\overline{11}$	$\overline{10}$	$\bar{9}$	$\bar{8}$	$\bar{7}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$
$\bar{3}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	$\bar{0}$	$\overline{11}$	$\overline{10}$	$\bar{9}$	$\bar{8}$	$\bar{7}$	$\bar{6}$	$\bar{5}$	$\bar{4}$
$\bar{4}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	$\bar{0}$	$\overline{11}$	$\overline{10}$	$\bar{9}$	$\bar{8}$	$\bar{7}$	$\bar{6}$	$\bar{5}$
$\bar{5}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	$\bar{0}$	$\overline{11}$	$\overline{10}$	$\bar{9}$	$\bar{8}$	$\bar{7}$	$\bar{6}$
$\bar{6}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	$\bar{0}$	$\overline{11}$	$\overline{10}$	$\bar{9}$	$\bar{8}$	$\bar{7}$
$\bar{7}$	$\bar{7}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	$\bar{0}$	$\overline{11}$	$\overline{10}$	$\bar{9}$	$\bar{8}$

$\bar{8}$	$\bar{8}$	$\bar{7}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	$\bar{0}$	$\overline{11}$	$\overline{10}$	$\bar{9}$
$\bar{9}$	$\bar{9}$	$\bar{8}$	$\bar{7}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	$\bar{0}$	$\overline{11}$	$\overline{10}$
$\overline{10}$	$\overline{10}$	$\bar{9}$	$\bar{8}$	$\bar{7}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	$\bar{0}$	$\overline{11}$
$\overline{11}$	$\overline{11}$	$\overline{10}$	$\bar{9}$	$\bar{8}$	$\bar{7}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	$\bar{0}$

The edges in the graph $\Gamma(Z_{12})$ are determined by using the Table 3 as in Example 3.2.

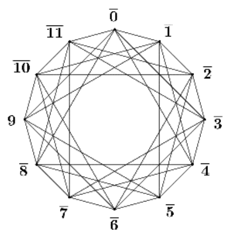

Fig.2: The graph $\boldsymbol{\Gamma(Z_{12})}$

SOME PROPERTIES OF THE IDEMPOTENT CAYLEY GRAPH

In this section, it is shown that the Idempotent Cayley graph $\Gamma(Z_n)$ is connected, Hamiltonian. Also it is bipartite for $n = 2^{\alpha}$, $\alpha > 1$ an integer.

Theorem 4.1: Let $n > 1$ be an integer. The Idempotent Cayley graph $\Gamma(Z_n)$ is a connected graph.

Proof: Let $\bar{r}, \bar{s} \in Z_n$ be any two vertices of the graph $\Gamma(Z_n)$ and let $r > s$. Then $0 \le s < r \le n-1$. For $0 \le s < r \le n-1$, $\overline{s+1} - \bar{s} = \bar{1} \in \mathbb{I}$, $\overline{s+2} - \overline{(s+1)} = \bar{1} \in \mathbb{I}, \dots, \overline{r-1} - \bar{r} = \bar{1} \in \mathbb{I}$.

So $(\bar{s}, \overline{s+1}), (\overline{s+1}, \overline{s+2}), \dots, (\overline{r-1}, \bar{r})$ are all edges of $\Gamma(Z_n)$ and $\bar{s} \longrightarrow \overline{s+1} \longrightarrow \overline{s+2} \longrightarrow \cdots \longrightarrow \overline{r-1} \longrightarrow \bar{r}$ is a path connecting $\bar{s}$ and $\bar{r}$. Thus the graph $\Gamma(Z_n)$ is a connected graph.

Theorem 4.2: The Idempotent Cayley graph $\Gamma(Z_n)$ of the ring $(Z_n, \oplus, \odot)$ is Hamiltonian.

Proof: Let $n \ge 1$ be an integer. Clearly $\bar{1} \in \mathbb{I}$. For an integer r, $0 \le r \le n-1$, $\overline{r+1} - \bar{r} = \bar{1} \in \mathbb{I}$, so that $(\bar{r}, \overline{r+1})$ is an edge of $\Gamma(Z_n)$. This shows that $\bar{0} \longrightarrow \bar{1} \longrightarrow \bar{2} \longrightarrow \cdots \longrightarrow \bar{r} \longrightarrow \overline{r+1} \longrightarrow \cdots \longrightarrow \overline{n-1} \longrightarrow \bar{0}$______________(1) is a closed path passing exactly once through every vertex of the graph $\Gamma(Z_n)$ and hence it is a Hamilton cycle in $\Gamma(Z_n)$. So the graph $\Gamma(Z_n)$ is Hamiltonian.

Definition 4.3: The cycle $\bar{0} \longrightarrow \bar{1} \longrightarrow \bar{2} \longrightarrow \cdots \longrightarrow \bar{r} \longrightarrow \overline{r+1} \longrightarrow \cdots \longrightarrow \overline{n-1} \longrightarrow \bar{0}$ is called the **outer Hamailton cycle** of the graph $\Gamma(Z_n)$.

Theorem 4.4: Let $n = p^m$, where p is a prime and $m \ge 1$ is an integer. Then the Idempotent Cayley graph $\Gamma(Z_n)$ contains only the outer Hamilton cycle $(\bar{0}, \bar{1}, \bar{2}, \dots, \overline{n-2}, \overline{n-1}, \bar{0})$.

Proof: Let $n = p^m$, where p is a prime and $m \ge 1$ is an integer. Since n is a power of a single prime, the number of idempotent elements in $(Z_{p^m}, \oplus, \odot)$ is $2^1 = 2$ and they are $\bar{0}$ and $\bar{1}$. So $\mathbb{I} = \{\bar{1}, \overline{(p^m - 1)}\}$. For any vertex $\bar{r}$, $0 \le r \le p^m - 1$ of the graph $\Gamma(Z_n)$, $\bar{r} \oplus \bar{1}$ and $\bar{r} \oplus$

$(\overline{p^m} - \bar{1})$ are the only two vertices adjacent to $\bar{r}$. But $\bar{r} \oplus (\overline{p^m - 1}) = \overline{r-1}$, since $\overline{p^m} = \bar{0}$. So $(\bar{r}, \overline{r+1})$ and $(\overline{r-1}, \bar{r})$ are the only two edges through $\bar{r}$, for all r, $0 \le r \le p^m - 1$.

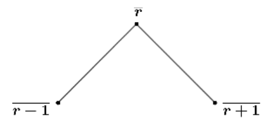

Fig.3: The connected path from $\overline{r-1}$ to $\overline{r+1}$

This shows that $\Gamma(Z_{p^m})$ contains only the outer Hamilton cycle

$(\bar{0}, \bar{1}, \bar{2}, \dots, \overline{r-1}, \bar{r}, \overline{r+1}, \dots, \overline{p^m - 1}, \bar{0})$.

Example 4.5: Consider the graph $\Gamma(Z_9)$. Here $9 = 3^2$, 3 is a prime. So $(Z_9, \oplus, \odot)$ has $2^1 = 2$ idempotent elements, namely $\bar{0}$ and $\bar{1}$. So $I = \{\bar{0}, \bar{1}\}$, $I^* = \{\bar{1}\}$, $(I^*)^{-1} = \{\overline{9-1}\} = \{\bar{8}\}$ and $\mathbb{I} = \{\bar{1}, \bar{8}\}$. By Theorem 4.4, the graph $\Gamma(Z_9)$ is the outer Hamilton cycle as shown in the Fig.4.

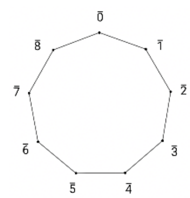

Fig.4: The graph $\Gamma(Z_9)$

Theorem 4.6: If $n = 2^\alpha$, $\alpha > 1$, then the graph $\Gamma(Z_n)$ is a bipartite graph.

Proof: Let $n = 2^\alpha$, $\alpha > 1$ an integer. As in Theorem 4.4, one can see that $\mathbb{I} = \{\bar{1}, \overline{(2^\alpha - 1)}\}$ and the graph $\Gamma(Z_n)$ consists of only the outer Hamilton cycle $(\bar{0}, \bar{1}, \bar{2}, \dots, \overline{r-1}, \bar{r}, \overline{r+1}, \dots, \overline{2^\alpha - 1}, \bar{0})$.

This graph is a bipartite graph with bipartition $A \times B$, where $A = \{\bar{0}, \bar{2}, \bar{4}, \dots, \overline{2^\alpha - 2}\}$ and $B = \{\bar{1}, \bar{3}, \bar{5}, \dots, \overline{2^\alpha - 1}\}$.

For example, when $n = 16 = 2^4$, $A = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}, \overline{12}, \overline{14}\}$ and $B = \{\bar{1}, \bar{3}, \bar{5}, \bar{7}, \bar{9}, \overline{11}, \overline{13}, \overline{15}\}$. The graph $\Gamma(Z_{16})$ and its bipartite version are given in Fig.5.

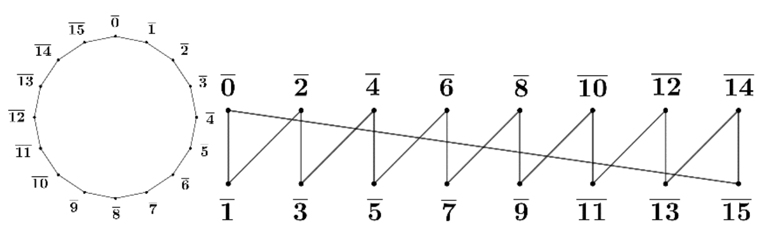

Fig.5: The graph $\Gamma(Z_{16})$ and its bipartite version

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