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The Lipschitz K(t) – Function for the Initial Value Problem (i.v.p) of an Ordinary Differential Equation (o.d.e) in One Dimension

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ABSTRACT

An elaborate but brief proof of the existence and uniqueness of the solution of an i.v.p for an o.d.e. is given. Because in both proofs the same initial condition is used, a considerable simplification of the K(t) Lipschitz function takes place. If the solution of the o.d.e. is known then the maximum interval of existence of the o.d.e. can be found again.

Keywords: Ordinary differential equation, initial value problem, K(t) Lipschitz function, maximum interval of continuation.

INTRODUCTION - MAIN THEME

For the i.v.p. of an o.d.e., in one dimension we can write:

$$y' = f(t,y),\ y(t_0)=y_0 \tag{1}$$

where $t \in IR$, $y \in IR$

The formal solution of (1) is:

$$y(t)=y_0+\int_{t_0}^{t} f(t', y(t'))dt' \tag{2}$$

which can be represented by the sequence of functions

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t}(t', y_n(t')dt', n \in IN, y_n(t_0) = y_0 \tag{3}$$

Furthermore, we suppose that the function f(t,y) is Lipschitz with respect to its second variable, i.e. [2]

$$|f(t,y) - f(t,x)| \leq K(t)\ |y\text{-}x|\ K(t)>0 \tag{4}$$

Which is a condition in between continuity and differentiability with respect to the second variable [3]

A Lipschitz function is continuous; a differentiable function is Lipschitz
Note: {a Lipschitz function f(t,x) not to be confused with the K(t)-Lipschitz function which has a similar name but different meaning}.

For the proof of the existence, we subtract (3) and (2) and we are using (4), by which:

$$|y_{n+1}(t) - y(t)| \leq \int_{t_0}^{t} K(t')\, |y_n(t') - y(t')| dt' \tag{5}$$

Defining:

$$M_n = \max_{[t_0,t]} |y_n(t') - y(t')| \tag{6}$$

We get by (5)

$$|y_{n+1}(t) - y(t)| \leq M_n \int_{t_0}^{t} K(t') dt' \tag{7}$$

Since if $a(t') \leq b(t')\ \forall t' \in |t_0, t| \Rightarrow \max\limits_{[t_0,t]} a(t') \leq \max\limits_{[t_0,t]} b(t')$

and since

$$\max_{[t_0,t]} \int_{t_0}^{t'} K(t'') dt'' = \int_{t_0}^{t} K(t') dt'\ (K(t) > 0)$$

we obtain by (7)

$$M_{n+1} \leq \wedge\, M_n \tag{8}$$

where $\wedge = \int_{t_0}^{t} K(t') < 1$. Since K(t) is positive, $\wedge\,(t_0, t)$ is positive and increasing that can reach the value 1 for t_1 large enough i.e. $\wedge\,(t_0, t_1) = 1$.

Limiting $t: t_0 \leq t < t_1$ we get always $\wedge\,(t, t_0) < 1$.

Noting that

$$M_0 = \max_{[t_0,t]} |y_0 - y(t)|$$

By (8), we easily obtain $M_n \leq M_o \wedge^n$

Since

$$0 \leq \wedge < 1\ (t_0 \leq t < t_1)$$

we obtain $\lim\limits_{n \to \infty} M_n = 0$

Which finally implies that the sequence of functions $y_n(t)$ (3) tends to the formal solution (2). For the proof of the uniqueness of the solution one defines the second solution.

$$x(t) = y_0 + \int_{t_0}^{t} f\big(t', x(t')\big)dt' \tag{9}$$

Subtracting (9) by (2) and defining

$$M(t)= \max_{[t_0,t]} |y(t') - x(t')| \tag{10}$$

and using the same process as in the existence proof one gets $M \leq M \wedge = M \int_{t_0}^{t} K(t')dt'$ for $t_0 \leq t < t_1$ ($\wedge < 1$)

Since M is non-negative and $\wedge < 1$, the only way for the inequality $M \leq M \wedge$ to be satisfied is M=0, which implies x(t)=y(t) (uniqueness).

For the Lipschitz condition from the above proof, we have

$$|f(t, y(t)) - f(t, x(t))| \leq K(t)|\, x(t) - y(t)| \tag{11}$$

Where K(t) has an "obvious solution":

$$K(t) = \frac{|f(t,y(t)) - f(t,x(t))|}{|y(t) - x(t)|} \tag{12}$$

And

$$y(t) = y(y_0,t),\ x(t) = y\ (x_0, t).$$

Since in the existence and uniqueness theories we have taken $x_0 = y_0$, K(t) finally tends to $\widetilde{K}(t)$ where

$$\widetilde{K}(t) = \lim_{x_0 \to y_0} \frac{|f(t,y_{(t)} - f(t,x_{(t)}|}{|y(t) - x(t)|} \tag{13}$$

Taking the de l' Hospital limit of the variables

$$x_0, y_0, y_0 \to x_0 \text{ we get } \widetilde{K}(t) = \frac{\left|\frac{\partial f}{\partial y}\right|\left|\frac{\partial y}{\partial y_0}\right|}{\left|\frac{\partial y}{\partial y_0}\right|} = \left|\frac{\partial f}{\partial y}\right|$$

Which finally gives:

$$\widetilde{K}(t) = \left|\frac{\partial f}{\partial y}(t, y(t)\right| \tag{14}$$

Now for the Λ-condition ($\wedge < 1$) we have $\widetilde{\wedge}(t, t_0) = \int_{t_0}^{t} \widetilde{K}(t')dt' < 1$ ($t_0 \leq t < t_1$) where t_1 is defined by

$$\widetilde{\Lambda}\,(t_0, t_1) = \int_{t_0}^{t_1} \widetilde{K}\,(t')dt' = 1 \tag{15}$$

In the following lines some examples of o.d.e's (i.v.p) in one dimesion are given to display the above ideas:

Let us consider the i.v.p:

$$\text{(A)}\ y' = y^2, y(t_0) = y_0, t_0 > 0, y_0 > 0 \tag{16}$$

i.e. f(t,y)=y^2 which has the solution

$$y(t) = \frac{y_0}{1-y_0(t-t_0)} = \frac{1}{(1/y_0)+t_0-t}\ t \geq t_0 \tag{17}$$

And for which

$$\widetilde{K}(t) = \left|\frac{\partial f}{\partial y}\right| = 2|y| :\ t_0 \leq t < t_0 + \frac{1}{y_0}$$

According to the theorem, the solution exists and is unique from t_0 to t_1 where

$$\widetilde{\Lambda}\,(t_0, t_1) = \int_{t_0}^{t_1} \widetilde{K}\,(t)dt = -2\ln\left[\frac{t_0+(1/y_0)-t_1}{1/y_0}\right] = 1 \tag{18}$$

i.e.

$$t_0 < t_1 = t_0 + \frac{1}{y_0}\left(1 - \frac{1}{\sqrt{e}}\right) < t_0 + \frac{1}{y_0} \tag{19}$$

Actually, we have:

$$t_1 = t_0 + \frac{\kappa}{y_0}\ \kappa = 1 - \frac{1}{\sqrt{e}}$$

and

$$y_1 = \frac{y_0}{1 - y_0(t_1 - t_0)} = \frac{y_0}{1-\kappa}$$

Considering again the point (t_1, y_1) as the initial point (t_0, y_0) we get for the (t_2, y_2)

$$t_2 = t_1 + \frac{\kappa}{y_1} = t_0 + \frac{\kappa}{y_0} + \kappa\frac{(1-\kappa)}{y_0} \text{ and } y_2 = \frac{y_0}{(1-\kappa)^2}$$

Going on with this process we get finally

$$t_n = t_0 + \frac{1}{y_0}[1-(1-\kappa)^n] \quad (20a)$$

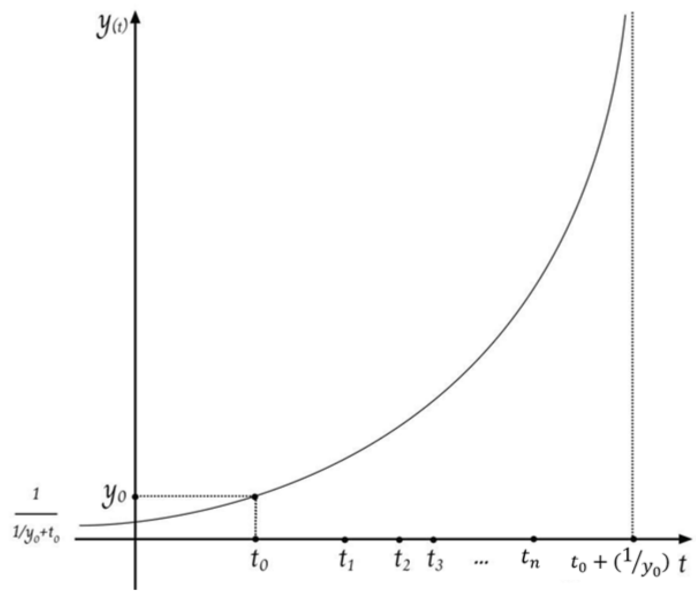
and

$$y_n = \frac{y_0}{(1-\kappa)^n} \quad (20b)$$

having done a continuation of the solution from t_1 to t_n where

$$t_0 < t_1 < t_n < t_0 + \frac{1}{y_0}$$

Since $0 < \kappa = 1 - \frac{1}{\sqrt{e}} < 1$ we have $\lim_{n\to\infty} t_n = t_0 + \frac{1}{y_0}$ and $\lim_{n\to\infty} y_n = +\infty$ (See Figure I)



We see that the whole interval of definition $\left(t_0, t_0 + \frac{1}{y_0}\right)$ of the function $y(t) = \frac{y_0}{1-y_0(t-t_0)}$ is reproduced if we successively use

$\widetilde{\Lambda}(t_0,t_1) = 1, \widetilde{\Lambda}(t_2,t_1) = 1, \ldots \widetilde{\Lambda}(t_n,t_{n-1}) = 1$, i.e. that the $\widetilde{\Lambda}$-condition $\widetilde{\Lambda} = \int_{t_0}^{t} \widetilde{K}(t')dt' < 1$ is exact if applied again and again to the new points found by definition. However, there is also a second lesson from this application, since $(t_1 - t_0) > (t_2 - t_1) > (t_3 - t_2) > \ldots\ldots$, the interval of integration allowed by the Lipschitz condition ($\widetilde{\Lambda}-$ condition) is smaller, the higher the valuer of y near the singularity $t = t_0 + \frac{1}{y_0}$, denoting that there is difficulty in its calculation.

Let's also see the i.v.p of the o.d.e

$$(B)\; y' = \frac{1}{2y}, y(t_0) = y_0, t_0 > 0, y_0 > 0 \quad (21)$$

Which has the solution

$$y^2 = y_0^2 + (t - t_0) \tag{22}$$

For convenience let us also consider $y_0^2 > t_0$

Since $\widetilde{K}(t) = \frac{1}{2y^2}$ and from $\widetilde{\Lambda}(t_1, t_0) = \int_{t_0}^{t_1} \widetilde{K}(t)dt$ we get $t_1 = t_0 + \lambda y_0^2$ where $\lambda = e^2 - 1$

$$y_1^2 = y_0^2 + (t_1 - t_0)$$

using the same process of continuation $t_0 \longrightarrow t_1 \longrightarrow t_2$, we find

$$(t_n - t_0) = y_0^2[(1 + \lambda)^n - 1] \tag{23a}$$

$$y_n^2 = y_0^2(1 + \lambda)^n \tag{23b}$$

By which we get $\lim_{n\longrightarrow\infty} t_n = +\infty$ and $\lim_{n\longrightarrow\infty} y_n = +\infty$ in accordance to the exact solution. (22)

(C) Another o.d.e is $y' = p(t)y + q(t) = f(t, y)$ (24)

With $y(t_0) = y_0, t_0 > 0, y_0 > 0$ with solution

$$y(t) = y_0 e^{\tilde{p}(t)} + e^{\tilde{p}(t)} \int_{t_0}^{t} q(t')\, e^{-\tilde{p}(t')} dt', \tilde{p}(t) = \int_{t_0}^{t} p(t')dt' \tag{25}$$

with $\widetilde{K}(t) = \left|\frac{\partial f}{\partial y}\right| = |p(t)|$.*

* provided that the functions p(t) and q(t) do not have singularities in the interval $[t_0, +\infty]$

From the $\widetilde{\Lambda}$ -condition we have $\widetilde{\Lambda}(t_1, t_0) = \int_{t_0}^{t_1} |p(t)|\, t = 1$

We define the function $P_a(t) = \int_{t_0}^{t} |p(t')| dt' > 0$ which is positive and increasing.

By the successive implementation of the $\widetilde{\Lambda}$-condition we have

$P(t_1) - P(t_0) = 1, P_a(t_2) - P_a(t_1) = 1, \ldots..$

By which we find

$$t_n = P_a^{-1}(n + P_a(t_0)) \tag{26}$$

Since $P_a(t)$ is positive and increasing, so is also ${P_a}^{-1}(t)$ by which we have $\lim_{n\to\infty} t_n = \lim_{n\to\infty} {P_a}^{-1}(n + P_a(t_0) = +\infty$, an information which comes without the exact knowledge of y(t).

CONCLUSION

The repeated implementation of Lipschitz - $\widetilde{\Lambda}$ - condition: $\widetilde{\Lambda}(t_1, t_0) = \int_{t_0}^{t_1} \widetilde{K}(t)dt = 1$, $\widetilde{\Lambda}(t_2, t_1) = 1$, ….. $\widetilde{\Lambda}(t_n, t_{n-1}) = 1$ and the subsequent finding of the sequence $\{t_n\}\, n = 0,1,2, \dots \infty$ can lead to the true interval of definition $[t_0, t_f]$ of the function $y(t)$ $(y'(t) = f(t, y(t)), y(t_0) = y_0)$ by imposing $t_f = \lim_{n\to\infty}\{t_n\}$ sometimes without the knowledge of the solution $y(t) = y(y_0, t)$. Usually, $\lim_{n\to\infty} t_n = +\infty$ unless there is somewhere a moving or a constant singularity.

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