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Complex Analytical Fiber Over Non-Compact Riemann Surfaces

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ABSTRACT

In this paper we explained the Complex Analytical Fiber Over Non-Compact Riemann Surface. We aimed to a relation between Complex analytic, Riemann Surfaces and Riemann Hilbert problems. The Compact Complex Surface S to a Riemann surface B such that the general fiber of$f$ is a Riemann surface of genus g. and singular for Riemann Hilbert problems.

Keywords: Fiber Product, Complex, Non- Compact, Analytic, Riemann, Surfaces, Diffeomorphism, Singular.

INTRODUCTION

A function ***f*** from a set ***A*** to a set ***B*** is a rule of correspondence that assigns to each element in ***A*** one and only one element in ***B***. A function ***f*** is thought of in this way, it is often referred to as a mapping, or transformation. The image of a point ***z*** in the domain of definition ***S*** is the point. ***w = f(z)***, and the set of images of all points in a set ***T*** that is contained in ***S*** is called the image of ***T***. [5], p73.The image of the entire domain of definition ***S*** is called the range of ***f***. We often think of a function as a rule or a machine that accepts inputs from the set ***A*** and returns outputs in the set ***B***. In elementary calculus we studied functions whose inputs and outputs were real numbers. Such functions are called real-valued functions of a real variable. Here we began our study of functions whose inputs and outputs are complex functions are simply generalizations of well-known functions from calculus. A function ***f*** from a set ***A*** to a set ***B*** is a rule of correspondence that assigns to each element in ***A*** one and only one element in ***B***. [17], p3

COMPLEX ANALYTICAL OF FIBER PRODUCT

A complex function **w = *f* (z)** is said to be analytic at a point ***z*** if ***f*** is differentiable at ***z*** and at every point in some neighborhood of ***Z***. [13], p73.

We already defined analytic functions between Riemann surfaces and also from some analytic structure and ask our- selves whether this data represents an integrable system. [18], p179

Even though the requirement of differentiability is a stringent demand, there is a class of functions that is of great importance whose members satisfy even more severe requirements. These functions are called analytic functions. [2], p145

Definition (2.1)

An elliptic function with periods K and L two $\mathbb{R}$ -independent complex numbers are a meromorphic function $\ell e : C \to C$ such that $\ell(z) = \ell\ (z + mK + nL)$ for all $m, n \in Z$. We now compute, by the fundamental theorem of calculus [1], P2.

FIBER PRODUCT OF RIEMANN SURFACES

Now, we introduce the concept of fiber product over Riemann surfaces. also, we explore the irreducible components associated to the fiber product and we defined the normal fiber product is true for all maps from non-compact Riemann surfaces into C, C^*, the Riemann sphere or complex tori. We devoted to study the fiber product at the level of connected Riemann surfaces. Let S_0, S_1, S_2 , be connected and not necessarily compact Riemann surfaces.

Definition (3.1)

Let us fix three Riemann surfaces, S_0, S_1 and S_2, and two surjective holomorphic maps $\beta_1 : S_0 \to S_1$ and $\beta_2 : S_2 \to S_0$. The fiber product associated to the pairs (S_1, β_1) and (S_2, β_2) is defined as

$$S_1 \times_{(\beta_1, \beta_2)} S_2 := \{(z_1, z_2) \in S_1 \times S\,2 : \beta_1\,(z_1) = \beta_2(\,z_2)\} \qquad (1)$$

Endowed with the topology induced by the product topology of $S_1 \times S_2$. There is associated a natural continuous map $\beta: S_1 \times (\beta_1, \beta_2)\ S_2 \to S_0$, such that:

$$(1)\ \beta = \beta_1 \circ \pi_1 = \beta_2 \circ \pi_2 \qquad (2)$$

Where $\pi_j: S_1 \times_{(\beta_1, \beta_2)} S_2 := S_j$ is the projection map $\pi_j(z_1, z_2) = z_j$, for $j \in \{1, 2\}$.

The fiber product of the pairs (S_1, β_1) and (S_2, β_2) enjoys the following universal property.

Definition (3.2)

The union of all the irreducible components is said the normal fiber product, which is denoted by $S_1 \times_{(\beta_1, \beta_2)} S_2$ this is the normalization of the fiber product $S\ S_1 \times_{(\beta_1, \beta_2)} S_2$ when it is considered as a complex algebraic variety

Proposition (3.3)

If β_1 and β_2 both have finite degrees, then the number of irreducible components of the fiber product of the two pairs (S_1, β_1) and (S_2, β_2) is at most the greatest common divisor of the degrees of β_1 and β_2. [9], p3 the concept of a conformal isomorphism. Note that any conformal isomorphism has a conformal inverse. The Riemann surfaces with conformal structures induced by (Q,z) and (n,z), respectively, do not have equivalent conformal structures but are conformally isomorphic [8] ,p4.

A RIEMANN SURFACE

Is a two-dimensional, connected, Haus topological manifold M with a countable base for the topology and with conformal transition maps between charts so complete analytic function" had been introduced previously for the collection of all function elements obtained via analytic continuation from however, from now on we shall use this term exclusively in the new sense. Next, we turn to the special case where the Riemann surface is compact [3]P3.

COMPACT RIEMANN SURFACES

Corollary (5.1)

Let $\boldsymbol{M}$ be a compact Riemann surface and ***f: M ……+ N*** an analytic nonconstant map. Then $\boldsymbol{f}$ is onto and $\boldsymbol{N}$ is compact.

Corollary (5.2)

Let $\boldsymbol{M}$ be a Riemann Surface. Then the following properties hold:

1. if $\boldsymbol{M}$ is Compact, then every holomorphic Function on $\boldsymbol{M}$ is constant.
2. Every nonconstant meromorphic function on a compact Riemann surface is onto $\boldsymbol{C}$.
3. If $\boldsymbol{f}$ is a nonconstant holomorphic Function on a Riemann Surface $\boldsymbol{M}$,

then If $\boldsymbol{I}$ attain neither a local maximum nor a positive local minimum on $\boldsymbol{M}$. The analytical ingredient in this proof consists of the uniqueness and open mapping theorems as well as the removability theorem: the first two are reduced to the same properties in charts which then require.

Let $\boldsymbol{\Sigma}$ be a bordered Riemann surface with genus g and analytic boundary components. [15], p157

Definition (5.3)

Two atlases $\boldsymbol{A}$ and $\boldsymbol{B}$ are called equivalent, if $\boldsymbol{A \cup B}$ is a holomorphic atlas. An equivalence class of atlases is called holomorphic structure. A Riemann Surface is a Haus'd or pace X together with a holomorphic structure on $\boldsymbol{X}$. Furthermore, a compact Riemann surface is a Riemann surface $\boldsymbol{X}$, such that any open cover of $\boldsymbol{X}$ has a finite subcover. [13], p158

Theorem (5.4)

There is a bijective correspondence between the set of conformal equivalence classes of Compact Riemann surfaces and the set of birational equivalence classes of algebraic function fields in one variable.

Example (5.5)

The developing map of the canonical projective structure is the inclusion map $\boldsymbol{U} \rightarrow \boldsymbol{p_1}$ of the universal cover of $\boldsymbol{C}$. More generally, when the projective structure is induced by a quotient map $\boldsymbol{\pi : U \rightarrow C = U/A}$ like, then the developing map is the universal cover U →U of U and the monodromic group is A, the open set $\boldsymbol{U}$ is not simply connected and the developing map is a non trivial covering. Thus, the corresponding projective structure is not the canonical one. Similarly, the developing map of a quasi-Fuchsian group is the uniformization map of the corresponding quasi-disk and is not trivial; the projective structure is neither the canonical one, nor of Schottky type.

The uniformization theorem for Compact Riemann Surfaces is

Theorem (5.6)

Let $\sum_2$ be a Compact Riemann Surface of genus p. Then there exists a conformal diffeomorphism

$$f: \sum_1 \to \sum_2 \tag{3}$$

where $\sum_2$ is

1. a compact Riemann surface of the form H/Γ as in case $p \geq 2$
2. a compact Riemann surface C/M in case $p = 1.$
3. the Riemann sphere s^2 in case $p=0$.

The $\sum_1$ is always homeomorphic to one of the types occurring in the statement. We start with case (1): $p \geq 2.$ $\sum_1$ then is homeomorphic to a hyperbolic Riemann surface S. S thus carries a metric of constant negative curvature. A homeomorphism from $\sum_1$ to S can be deformed into a harmonic map

$$\mathbf{u} : \sum_1 \to \mathbf{S}. \tag{4}$$

u then has degree ± 1, since a homeomorphism has degree ±1 and the degree is not changed under homotopes. in fact, one easily verifies that there always exists a homeomorphism is: $S \to S$ of degree -1, and if the original homeomorphism had degree -1, its composition with $i0$ then has degree 1. Thus, we may always find a harmonic.

$$u : \sum_1 \to S \tag{5}$$

of degree 1. u then is a diffeomorphism. As before u induces a holomorphic quadratic differential ψ on $\sum_1$. We put $S = S_1,\ u =: u^1$ and the strategy now is to find a harmonic diffeomorphism $u^t : \sum_1 \to S_t$ onto a hyperbolic Riemann surface St with induced holomorphic quadratic differential ψ for all $t \in [0,1].$ For $t=0$ the map

$$u^0 : \sum_1 \to S_0 \tag{6}$$

Then is a conformal diffeomorphism, since the associated holomorphic quadratic differential vanishes. Putting $\sum_2 = S_1$ then finishes the proof in case (1). we are going to show that:

$$t_0 = inf\{t \in [0,1]: u^t, S_t \text{, exists for } all \ \geq \mathbf{t}\} = 0 \tag{7}$$

Again, the set over which the infimum is taken, is nonempty because it contains $t=1$. This set is again open by an implicit function theorem argument. The interesting point is closedness. We equip $\sum_1$ with an arbitrary smooth conformal Riemannian metric which we write again as $\lambda^2(z)dzdz$ in local coordinates, although its curvature need no longer be -1. The image metric again is denoted by $e^2(\ u^t)$ out and with u^t, we again associate the expressions

$$H(T) = \frac{e^2(\ u^t(z))}{\lambda^2\ (z)} u_z^t \pi \frac{t}{z} \tag{8}$$

$$L(T)=\frac{e^2(u^t(z))}{\lambda^2(z)}\pi_z^t u\frac{t}{z} \quad \textbf{(9)}$$

We have:

$$H(t)L(t)=t^2\frac{1}{\lambda^4}\psi\,\overline{\psi} \quad \textbf{(10)}$$

Differentiating write. t yields:

$$H(t)L(T)+H(t)L(t)=t^2\frac{1}{\lambda^4}\psi\,\overline{\psi}=\frac{2}{t}H(t)L(t) \quad \textbf{(11)}$$

$$\Delta \log H(t) = 2K_1 + 2\,(H(t) - L(t)) \quad \textbf{(12)}$$

since the curvature of S_t is -1. K_1 here denotes the curvature of $\sum_1$. Differentiating (2.10) write t and using it yields

$$\Delta= \frac{\dot{H}(t)}{H(t)}=2\frac{\dot{H}(t)}{H(t)}H(t)L(t)-\frac{2}{t}L(t) \quad (13)$$

Again, we must have $\Delta\frac{\dot{H}(t)}{H(t)}(z_1)\geq 0$ at appoint

$$\frac{\dot{H}(t)}{H(t)}(z_1) \quad \textbf{(14)}$$

achieves its minimum. Since $L(t)\geq 0$ by definition of L, we conclude

$$\dot{H}(t)(z) \geq 0 \quad \textbf{(15)}$$

for all **z.** Therefore

$$H(t)\leq H(L)$$

whenever

$$0< t <1. \quad \textbf{(16)}$$

we also know

$$0 \leq L(t) < H(t).$$

preceding notations

$$\Delta \log H(t) = 2K_1 \quad \textbf{(17)}$$

and consequently

$$\frac{\dot{H}(t)}{H(t)} = 0 \qquad (18)$$

thus $\frac{\dot{H}(t)}{H(t)}$ is a harmonic Function on the compact Riemann surface $\sum_1$, hence constant. This means: ***H(t)(z)=e(t)(z)***for some constant e.[6] ,p16

Corollary (5.7)
The universal cover of a compact Riemann surface is con- formally equivalent to s^2**, C** or the unit disk ***D***.

Remark (5.8)
The case ***p=1*** of the uniformization theorem can also be deduced from the Jacobi Inversion Theorem. [6], p17

NON-COMPACT RIEMANN SURFACES

Theorem (6.1)
The first cohomotopy group H^1***(X,*** $\mathcal{E}$***)*** vanishes for any Riemann Surface ***X***. The following is one of the most fundamental results for non-Compact Riemann Surfaces. It strengthens the Debleats lemma from local solvability of the ∂ equation to global solvability for non-compact Riemann surfaces.

Theorem (6.2)
Let ***S*** be a Non-Compact Riemann surface and $\omega \in \mathcal{E}^{0,1}(S)$. Then there is $g \in \mathcal{E}(S)$ such that:

$$\partial g = \omega. \qquad (19)$$

Corollary (6.3): Let ***S*** be a Non-Compact Riemann surface. Then $H^1(S, \mathcal{O}) = 0$.

Proof: Consider the following short exact sequence of sheaves

$$0 \to \mathcal{O} \hookrightarrow \mathcal{E} \xrightarrow{\vartheta} \mathcal{E}^{0,1} \longrightarrow 0 \qquad (20)$$

That the sequence is exact at $\mathcal{E}^{0,1}$ follows from the Dolbeault. Namely, if $p \in S$ and $\omega. \in \mathcal{E}_p^{(1,0)}$ is of the form $\omega = g\, \partial\, \bar{z}$ in coordinates (U, z) at p, then there is $f \in \mathcal{E}_p(U)$ such that $\partial f / \partial \bar{z} = g$ and hence $\partial f = \omega$ on ***U***. Since $H^1(S, \mathcal{E})$ vanishes, we have

$$H^1(S, \mathcal{O}) = \mathcal{E}^{0,1}(S)/\partial \mathcal{E}(S). \qquad (21)$$

It follows that $H^1(S, \mathcal{O}) = 0$ if and only if $\mathcal{E}^{0,1}(S) = \partial\text{-}\, \mathcal{E}(S)$.

It is a standard result of sheaf homology that $H^1(X, \vartheta) = 0$ if and only if $H^1(X, \vartheta) = 0$ for every open cover U of X Hence, we note that H^1 (X, €) if and only if H^1 ($\mathcal{U}$, $\mathcal{E}$) vanish for every open cover $\mathcal{U}$ of a Non-Compact Riemann Surface.

There are several results on the flexibility of holomorphic and meromorphic functions that are consequences.

Theorem (6.4)
On a non-compact Riemann surface every Mittag-Leffler distribution has a solution. [17], p18

The notion of divisors plays an important role in Riemann surface theory, most notably in the celebrated Riemann-Roch theorem for compact Riemann surfaces. A divisor on a Riemann surface $\boldsymbol{X}$ is a map $\boldsymbol{X \to Z}$ with discrete support. To each meromorphic function f: X → p_1 which is not identically zero, we associate a divisor ***(f): X → Z, x →*** $\boldsymbol{ord_x}$ ***f***. Here ord_xx f is the order of ***f*** at x, by which we mean, the divisor of a meromorphic function keeps track of the order of its zeros and poles. We say a divisor is principal if it is the divisor of some meromorphic function. It turns out that on Non-Compact Riemann Surfaces every divisor is the divisor of a meromorphic function. [15], p157

Theorem (6.5)
On a Non-Compact Riemann Surface every divisor is principal.

Corollary (6.6)
For a Non-Compact Riemann Surface ***S,*** $\boldsymbol{H^1}$ ***(S, D) = 0*** for any divisor ***D***.

Corollary (6.7)
Let S be a Non-Compact Riemann Surface. Then there exists a holomorphic 1-form on S which is nowhere vanishing.

Theorem (6.8)
Let S be a Non-Compact Riemann Surface and T be a discrete subset of S. Suppose ***C: T → C*** is an arbitrary map from T into ***C***. Then there is a holomorphic function ***f ∈ O(S)*** with ***f \ T = c***.

The following is a classical theorem of Behnke and Stein. [17],p19

Theorem (6.9)
Let S be a non-Compact Riemann S Surface and $\boldsymbol{C:\pi^1(S) \to C}$ be a group homomorphism. Then there is a holomorphic form $\boldsymbol{\omega \in \Omega(S)}$ with

$$\int\sigma\, \omega = c(\sigma) \text{ for each } \sigma \in \pi^1(S) \qquad \textbf{(22)}$$

Singular Riemann Surfaces. In this subsection, we shall explore some elements of the theory of the singular Riemann surfaces as the locus of singular points, irreducible components, isomorphism and the group of automorphism of a singular Riemann surface. The Poincare disc will be denoted by ***Δ.*** [3],p224

Definition (6.10)
A Singular Riemann surface is an one-dimensional Complex analytic surface ***S,*** such that for each point ***p*** of ***S*** there exists a neighborhood holomorphically equivalent to a subspace of the form

$$V_{n,m} = \{(z, w) \in \Delta \times \Delta : z^n = w^m\} \subset \Delta \times \Delta \tag{23}$$

for some integers ***n, m ≥ 1***.

If ***n = 1*** or ***m = 1***, then $V_{n,m}$ is holomorphically equivalent to ***Δ***. Now, if ***n, m ≥ 2*** and ***d ≥ 1*** is the greatest common divisor of n and m, then we write n = $\hat{d}$ n and ***m =*** $\hat{d}$ ***m***, where ***n,m ≥ 1*** are relatively prime integers. Thus, the space $V_{n,m}$ can be written as

$$V_{n,m} = \{z, w) \in \Delta \times \Delta : \prod_{k=0}^{d-1} z^n - \omega^k w^m = 0\} \tag{24}$$

and it is homeomorphic to a collection of d cones with common vertex at (0, 0) where ω is **a, d** primitive root of unity. In particular, if d = 1, then the space $V_{n,m}$ is holomorphically equivalent to Δ.

If ***d ≥ 2,*** then the point $p \in S$ is called singular, and the locus of singular points of S, denoted by ***Sing(S)*** is a discrete subset of S. It follows that each connected component R of ***S \ Sing (S)*** has structure of Riemann surface, and the points in Sing (S) define punctures on R. By adding these punctures, we obtain another Riemann surface R, containing ˜R, called an irreducible component of ***S***. If S has only one irreducible component, then it is called irreducible; otherwise, it is called reducible [18], P47.

Remark (6.11)
If S is a Compact singular Riemann Surface, then it follows the following properties:

1. The locus of singular points ***Sing (S)*** is a finite set.
2. The singular Riemann surface ***S*** has many finitely irreducible components.
3. Each irreducible component ***R*** of ***S*** is a Compact Riemann Surface and ˜R is also a Riemann surface having finite punctures.

Theorem (6.12)
The first homology group $H^1(X,\Sigma)$ vanishes for any Riemann surface X. The following is one of the most fundamental results for non-compact Riemann surfaces. It strengthens the Debleats lemma from local solvability of the ∂ equation to global solvability for non-Compact Riemann Surfaces.

Theorem (6.13)
let S be a non-compact Riemann surface and $\omega \in \Sigma^{0,1}$ (**S**). Then there is $g \in \Sigma$ (***S***) such that

$$\partial g = \omega. \tag{25}$$

Corollary (6.14)
Let S be a Non-Compact Riemann Surface. Then $H^1(S,\bar{O}) = 0$.

Proof
Consider the following short exact sequence of sheaves

$$0 \to \bar{O} \hookrightarrow \mathcal{E} \xrightarrow{\vartheta} \mathcal{E}^{0,1} \to 0 \quad (26)$$

That the sequence is exact at $\mathcal{E}^{0,1}$ follows from the Dolbeault. Namely, if $p \in S$ and $\omega \in \mathcal{E}_p^{(1,0)}$ is of the form ω=g ∂ ¯z in coordinates *(U, z)* at p, then there is $f \in \mathcal{E}_p(U)$ such that ∂f /∂ ¯z = g and hence ∂f = ω on U. Since $H^1(S,\mathcal{E})$ vanishes, we have

$$H^1(S,\bar{O}) = \mathcal{E}^{0,1}(S)/\partial\mathcal{E}(S). \quad (27)$$

It follows that H^1(S, Ō) = 0 if and only if $\mathcal{E}^{0,1}$(S) = ∂- $\mathcal{E}$ (S).

It is a standard result of sheaf homology that H^1 (X, ϑ) = 0 if and only if

H^1 (X, ϑ) = 0 for every open cover U of X Hence, we note that H^1 (X, €) if and only if H^1 ($\mathcal{U}$, $\mathcal{E}$) vanish for every open cover $\mathcal{U}$ of a non-compact Riemann Surface. There are several results on the flexibility of holomorphic and meromorphic Functions that are consequences.

Theorem (6.15): On a Non-Compact Riemann Surface every Mittag-Leffler distribution has a solution.

The notion of divisors plays an important role in Riemann surface theory, most notably in the celebrated Riemann-Roch theorem for compact Riemann surfaces. A divisor on a Riemann surface X is a map $X \to Z$ with discrete support. To each meromorphic **function** f: $X \to p_1$ which is not identically zero, we associate a divisor *(f)*: $X \to Z$, $x \to ord_x f$. Here $ord_x x f$ is the order of f at x, by which we mean, the divisor of a meromorphic function keeps track of the order of its zeros and poles [4], P73.

We say a divisor is principal if it is the divisor of some meromorphic function. It turns out that on non-Compact Riemann Surfaces every divisor is the divisor of a meromorphic function.

Theorem (6.16)
On a Non-Compact Riemann Surface every divisor is principal.

Corollary (6.17)
For a Non-Compact Riemann Surface S, H^1 $(S, D) = 0$ for any divisor D.

C**orollary (6.18)**
Let S be a non-compact Riemann surface. Then there exists a holomorphic 1-form on S which is nowhere vanishing. [9], p1

Singular Riemann Surfaces
In this subsection, we shall explore some elements of the theory of the singular Riemann surfaces as the locus of singular points, irreducible components, isomorphism and the group of a automorphism of a singular Riemann surface. The Poincare disc will be denoted by Δ. [3], p224

Definition (6.19)
A Singular Riemann Surface is an one-dimensional complex analytic1 surface S, such that for each point p of S there exists a neighborhood holomorphically equivalent to a subspace of the form

$$V_{n,m} = \{(z, w) \in \Delta \times \Delta: z^n = w^m \} \subset \Delta \times \Delta \qquad (28)$$

for some integers ***n, m ≥ 1.***

If $n = 1$ or $m = 1$, then $V_{n,m}$ is holomorphically equivalent to Δ. Now, if ***n, m ≥ 2*** and d ≥ 1 is the greatest common divisor of n and m, then we write $n = \hat{d}$ n and $m = \hat{d}$ **m**, wherê $n\hat{,}$ $m \geq 1$ are relatively prime integers. Thus, the space $V_{n,m}$ can be written as

$$V_{n,m} = \{z, w) \in \Delta \times \Delta: \prod_{k=0}^{d-1} z^n - \omega^k w^m = 0 \} \qquad (29)$$

and it is homeomorphic to a collection of d cones with common vertex at (0, 0) where ω is a d-th primitive root of unity. In particular, if d = 1, then the space $\mathbf{V_{n,m}}$ is holomorphically equivalent to Δ.

If d ≥ 2, then the point p ∈ S is called singular, and the locus of singular points of ***S***, denoted by ***Sing (S)*** is a discrete subset of S. It follows that each connected component ˜***R*** of ***S \ Sing(S)*** has structure of Riemann surface, and the points in ***Sing(S)*** define punctures on ˜***R***. By adding these punctures, we obtain another Riemann surface ***R,*** containing ˜***R,*** called an irreducible component of ***S***. If ***S*** has only one irreducible component then it is called irreducible; otherwise, it is called reducible.

Remark (6.20):
If S is a Compact singular Riemann surface, then it follows the following properties:
1. The locus of singular points ***Sing(S)*** is a finite set.
2. The singular Riemann surface S has many finitely irreducible components.
3. Each irreducible component R of ***S*** is a compact Riemann surface and ˜***R*** is also a Riemann surface having finite punctures. [16], p15

Definition (6.21)
Metric Surface is a Compact Riemann Surface equipped with a conformal Riemannian metric. The reader should be warned that this definition is not usually standard in the literature, and therefore, we shall employ it only in the present section. Let ***M*** be a metric surface with metric

$$\lambda_2(z)\, dz\, dz. \qquad \mathbf{(30)}$$

The inverse function $\xi(\zeta)$ is doubly periodic, with periods K and L given by

$$K = 2 \int 1\ 0\ dz \text{ and } L = \int 1/k\ 0\ dz. \qquad \mathbf{(31)}$$

In other words, ξ is determined by its values on the parallelogram with vertices ***{0, K, L, K + L}***.

Lemma (6.22)
If f, g are (sufficiently generic, *i.e*. the diagonal map $z \mapsto (f(z), g(z))$ is injective this is true for x and y in Eq. (28) elliptic functions with periods K, L and h is a third such function, then h is a rational function in f, g.

Proposition (6.23)
Given a point y_0 in a connected Riemann surface Y and a holomorphic function ψ0 defined on a neigh boarhound of y0, there are then a Riemann surface X, a holomorphic map $F: X \to Y$ without branch points, a point x_0 in $F-1\ (y_0\ )$ and a holomorphic function on X such that: ψ_0 can be analytically continued along a path γ in Y if and only if γ has a lift to X starting at x_0. The analytic continuation of ψ_0 along γ has ψ_1 equal to $F-1\ \gamma$ in a neighborhood of $\gamma\ (1)$.

The Riemann surface X is the 'Riemann surface of the function germ ψ_0, and the data about all possible analytic continuations of ψ_0 is encoded in the holomorphic map $F: X \to Y$ [12], p54

Theorem (6.24)
Let X be a compact Riemann surface. If $E = [D]$ is the holomorphic line bundle associated to a nontrivial effective divisor D on X, E is a holomorphic line bundle that admits a nontrivial global holomorphic section with at least one zero, then E is positive [16]P2.

THE RIEMANN-HILBERT PROBLEM

Our first proposition claims that if there is a solution of the Riemann-Hilbert problem, then there is also one which has at most $2g + m - 1$ zeros on Σ. The idea of the proof is the fact that the family of solutions of the Riemann-Hilbert problem with more than $2g + m - 1$ then there exists a solution with at most $2g + m - 1$. [10], p47

Proposition (7.1)
If there exists a solution $f \in A_K\alpha(\Sigma)$ of the Riemann-Hilbert problem for $\{\gamma_z\}z \in \partial\Sigma$, zeros.

Exterior adjoint RH problem: Search a function g analytic in G with $g\ (\infty) = 0$, continuous on the closure G, algebraically of a solution of the Riemann-Hilbert problem and let $f \in A_K\alpha(\Sigma)$ be a solution with the minimal number of zeros. Let F be the family of all solutions $f \in A_K\alpha(\Sigma)$ of the Riemann-Hilbert problem such that $f: \partial\Sigma \to C \setminus \{0\}$ is homotopic to $F_0 : \partial\Sigma \to C \setminus \{0\}$, that is, f and f0 have the same winding number over each boundary component of Σ we orient the boundary $\partial\Sigma$ coherently with the natural orientation on Σ. The number of zeros of a function $f \in A_K,\alpha(\Sigma)$ which is nonzero on $\partial\Sigma$ equals the winding number of f along $\partial\Sigma$. Hence each f ∈ F has the same number of zeros counting algebraically [8], p241.

DICTATION

We construct all the fibered complex surfaces for possible monodromic representations and orbifold moduli maps by pulling back from our universal degenerating family.

We let $D(B) = \{Q_1, \cdots, Q_s\}$ be a finite set of a compact Riemann surface B, and set $B^{(0)} = B \setminus D(B)$. For a fixed point $b_0 \in B^{(0)}$ we consider a representation of the fundamental group to the mapping class group of genus g ≥ 2

$$\mu{:}\pi_1\ (B^{(0)},\ b_0) \longrightarrow \Gamma_g \tag{32}$$

Theorem (8.1)

Let $f: S \to B$ be a proper surjective holomorphic map from a Compact Complex surface S to a Riemann surface B such that the general fiber of f is a Riemann surface of genus g. Let Sign S be the signature of the intersection form on $H^n(S, Q)$. If there exist a finite number of fiber germs (f, F_i), $F_i = f^{-1}l\ (P_i)\ (1 \leq i \leq, P_i \in B)$ and a local invariant $\sigma(f, F_i)$ *is* geometrically well-defined such that

$$Sign\ S = \sum_{i=0}^{\infty} \sigma(f, F_i), \tag{33}$$

then we say Sign S is localized and call $a\ (f, F_i)$ a local signature. Our problem here is to discuss how to formulate and also how to calculate the local signature. It should be formulated by depending on the nature of the general fiber of f. [4], p4

Remark (8.2)

The construction of the Fibered Complex Surface of genus $g \geq 2$ is analogous to that of the basic members of elliptic surfaces (basic elliptic surfaces, for short) due to nevertheless they are different in the following points.

1. Basic elliptic surfaces are assumed to have no multiple Fibers $m_0^1 b$, $m \geq 2$ given in while our fibered surfaces are admitted to have any singular fibers including multiple Fibers [13], P74.
2. Let $F\ (J, G)$ be the set of elliptic surfaces without multiple fibers which have J-invariant J and homological invariant G Then the basic elliptic surface in $F\ (J, G)$ is characterized as the unique member which admits a global section and other members in $F\ (J, G)$ are obtained from the basic elliptic surface by twisting defined in. the other hand, a fibered complex surface of genus $g \geq 2$ is uniquely determined by the data (μ, J) This difference essentially comes from the fact that an elliptic curve has translation automorphisms, while a Reimann surface of genus $g \geq 2$ has no such infinite automorphism.

Ricarte foliations on p^1 -bundles over Compact Riemann Surfaces. Let $n: P \to C$ be a P-bundle Over a Compact Riemann Surface C: P is a smooth surface and the fibers of T are rational, isomorphic to p^1. We recall some begiven when a Riemann- Hilbert problem needs to be considered on an algebraic variety, a localized Riemann- Hilbert problem need only be solved in the complex plane and the local solution can then be glued to the global Riemann-Hilbert solution on the variety. [13], p3

RESULTS

A Riemann surface B such that the general fiber of f is a Riemann surface of genus g. the Riemann sphere or complex tori. This article is devoted to study the fiber product at the level of connected Riemann surfaces. Let S_0, S_1, S_2 , be connected and not necessarily compact Riemann surfaces. for every open cover $\mathcal{U}$ of a non-compact Riemann Surface. There are several results on the flexibility of holomorphic and meromorphic Functions that are consequences. The solution of RHP Proof by analytic complex plan and Fiber product [10], p47.

CONCLUSIONS

Complex analytical Fiber Over non -Compact Riemann surface it is one of proof some analytic problems in Complex plane and product fiber to Riemann Surface and solution of Riemann Hilbert problems. Any Compact Riemann Surface X of genus $g > 1$ admits a positive holomorphic line bundle with no nontrivial holomorphic sections [18]. we consider some facts concerning holomorphic line bundles on compact Riemann surfaces. Riemann surface X has dimension 1, and a negative holomorphic line bundle on X has no nontrivial global holomorphic. After the above considerations, we consider the fact that a holomorphic line bundle is positive if and only if its degree is positive. The fiber product has been studied over compact Riemann surfaces. Whereas along this paper, we will focus on the fiber product over non-compact Riemann surfaces, inasmuch as of classification theorem of non-compact surfaces. [5], p3

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