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Comparative Analysis of Schemes with Movable Nodes for a Parabolic Equation

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ABSTRACT

The article considers an approximate analytical solution of a linear parabolic equation with initial and boundary conditions. Many problems in engineering applications are reduced to solving an initial-boundary value problem of parabolic type. There are various analytical, approximate-analytical and numerical methods for solving such problems. The most popular difference methods for solving an initial-boundary value problem of a parabolic equation are explicit, implicit and Crank-Nicolson schemes. Here, we consider methods for obtaining an approximate-analytical solution based on the movable node method and their comparative analysis of these schemes for specific test problems. A comparison of the exact and approximate solutions is made using specific examples.

Keywords: parabolic equation, approximate-analytical solution, moving nodes.

INTRODUCTION

Processes in hydrodynamics, heat transfer, boundary layer flow, elasticity, quantum mechanics and electromagnetic theory are modeled by differential equations. Only some of these equations can be solved by an analytical method. But the search for exact solutions, when they exist, is always necessary to better explain the modeled phenomenon. The search for an analytical solution gives an advantage for analyzing processes [1,2,3].

Analytical methods have a relatively low degree of universality for solving such problems. More universal are approximate analytical methods (projection, variational methods, the small parameter method, operational methods, various iterative methods) [4,5,6,7].

Comparative analysis for solving shifted boundary value problems is carried out based on the method of moving nodes [8,9,10]. The method combines the approximation of derivatives appearing in the equation, difference relations and obtaining an approximate analytical expression for the solution of the problem. In this case, we can obtain an approximate analytical solution to the problem, which is a hybrid of known methods. Note that obtaining an approximate analytical solution to differential equations is based on numerical methods. The nature of numerical methods also allows obtaining an approximate analytical expression for the solution of differential equations. For this purpose, the so-called "movable node" is introduced [8]. The aim of the study is a comparative analysis of various difference schemes for applying the method of moving nodes for a parabolic type of equation and is a continuation of the work [10]. Compare explicit, implicit and Crank-Nicholson for a mixed problem of a parabolic equation and provide test examples.

STATEMENT OF THE PROBLEM

Let us consider a one-dimensional differential equation of parabolic type in the domain Ω: (0<t<T, W<x<E)

$$\frac{\partial u}{\partial t} = A\frac{\partial^2 u}{\partial x^2} + f(x,t); \tag{1}$$

with initial

$$u(x, t=0,) = u^0(x). \tag{2}$$

and boundary conditions

$$u(x=W,t) = u_W(t) \;\; u(x=E,t) = u_E(t). \tag{3}$$

We assume that the solution to problem (1)-(3) exists and is unique.

For the numerical solution of problem (1)--(3) there are various difference schemes [11]. Let us consider various variants of difference approximation of a linear one-dimensional equation in space by a moving node.

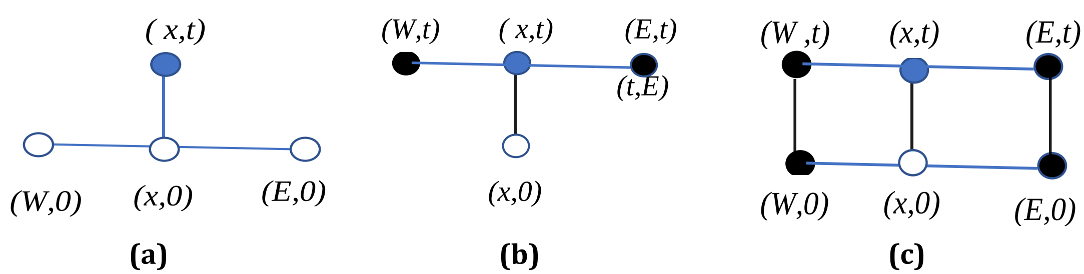


Fig 1: Template of a moving node

Fig. 1 shows templates with one moving node. Fig. 1 (a) corresponds to the template by the explicit scheme, Fig. 1 (b)– the implicit scheme, and Fig. 1 (c)– the Crank Nicholson scheme. In Fig. 1, the point (t, x) ϵ Ω refers to the moving node. Points (0, x), (t, W) and (t, E) are also movable nodes: point (0, x) moves only along the x-axis, point (t, W) moves along the left, and point (t, E) moves along the right boundary of the region.

SOLUTION BY MOVING NODES METHOD

Using one movable node, we can obtain a rough analytical representation of the solution to problem (1)-(3).

Let us denote by $U(x,t)$. an approximate analytical solution to the problem obtained using the movable node method and using the boundary and initial conditions. Let $(x,t)\in\Omega$ be an arbitrary moving point. We approximate equation (1) with an explicit scheme

$$\frac{U(x,t)-U^0(x)}{t}=A\frac{2}{E-W}\left(\frac{U^0{}_E(E)-U^0(x)}{E-x}-\frac{U^0(x)-U^0(W)}{x-W}\right)+f(x,t), \tag{4}$$

In (4) is an approximate analytical solution to the problem. When the point runs through, we obtain a solution in the region under consideration. From (3) we obtain

$$U(x,t)=U^0(x)+A\frac{2t}{E-W}\left(\frac{U^0{}_E(E)-U^0(x)}{E-x}-\frac{U^0(x)-U^0(W)}{x-W}\right)+tf(x,t), \tag{5}$$

If we perform approximation using the implicit scheme, we have

$$\frac{U(x,t)-U^0(x)}{t}=A\frac{2}{E-W}\left(\frac{U_E(t)-U(x,t)}{E-x}-\frac{U(x,t)-U_W(t)}{x-W}\right)+f(x,t), \tag{6}$$

By solving this equation, we obtain an approximate analytical solution to the problem in the case of an implicit scheme

$$U(x,t)=\frac{(E-x)(x-W)}{2At+(E-x)(x-W)}U^0(x)+\frac{2At\left[U_E(t)(x-W)+U_W(t)(E-x)\right]}{2At+(E-x)(x-W)}+ \\ \frac{(E-x)(x-W)t}{2At+(E-x)(x-W)}f(x,t). \tag{7}$$

The Crank-Nicholson scheme has the form:

$$\frac{U(x,t)-U^0(x)}{t}=\sigma A\frac{2}{E-W}\left(\frac{U_E(t)-U(x,t)}{E-x}-\frac{U(x,t)-U_W(t)}{x-W}\right)+ \\ (1-\sigma)A\frac{2}{E-W}\left(\frac{U^0(E)-U^0(x)}{E-x}-\frac{U^0(x)-U^0(W)}{x-W}\right)+f(x,t), \tag{8}$$

From here we determine the approximate analytical solution using the Crank-Nicholson scheme:

$$U(x,t)=\frac{(E-x)(x-W)}{D}U^0(x)+B\frac{(x-W)U_E(t)-(E-x)U_W(t)}{D}+ \\ C\frac{(x-W)(U^0(E)-U^0(x))-(E-x)(U^0(x)-U^0(W))}{D}+\frac{(E-x)(x-W)t}{D}f(x,t), \tag{9}$$

In expression (9), the notation is introduced

$$B = \frac{2\sigma A t}{E - W}; \quad C = \frac{2(1-\sigma) A t}{E - W}; \quad D = (E - x)(x - W) + B(E - W).$$

Let's consider test examples.

EXAMPLES OF PROBLEM SOLVING

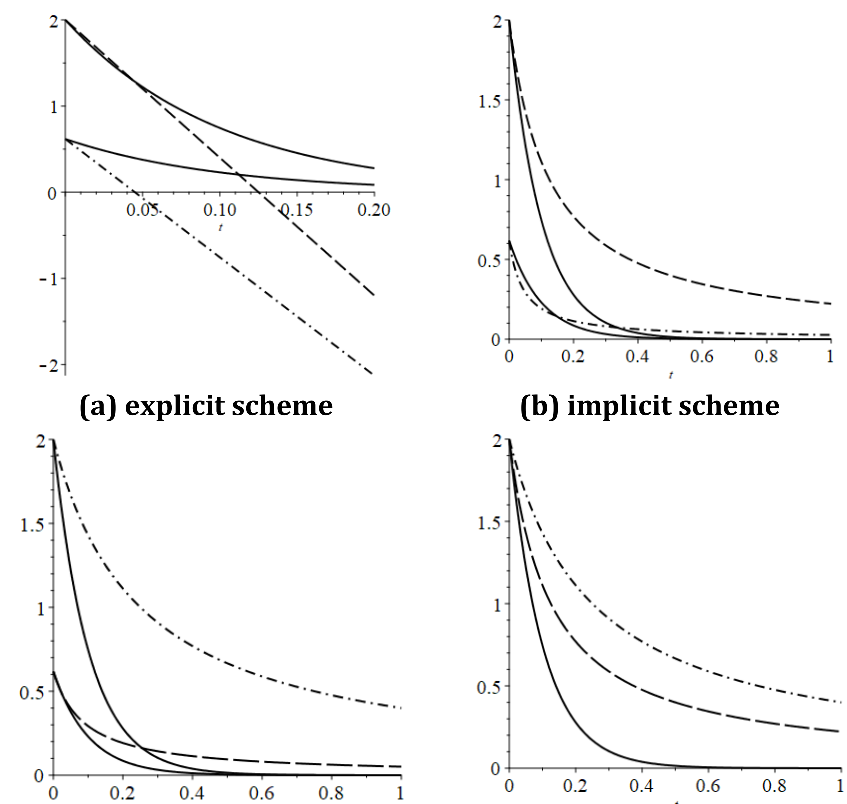
First Problem

Consider equation (1) in the domain Ω = {(x,t) |0 ≤ x ≤ 1, 0 ≤ t ≤ T} with initial and boundary conditions

$$U^0(x) = 2\sin\frac{\pi x}{h}, \;\; U_W(t) = 0, \, U_E(t) = 0,$$

and with the right-hand side of the form: $f(x,t) = 0.$ The exact solution of this problem

$$U(x,t) = 2\sin\left(\frac{\pi x}{h}\right)\exp\left(-\frac{\pi^2 A}{h^2} t\right).$$



(a) explicit scheme

(b) implicit scheme

(c) Crank-Nicholson scheme

(d) comparisons implicit and according to the Crank-Nicholson scheme олсона

Fig. 1: Comparisons of solutions for problem 1.

Fig. 1 shows a comparison of the considered schemes for some sections. In these graphs, solid lines correspond to the exact solution. Fig. 1(a) shows that the method of a moving node constructed according to the explicit one can be used only in the initial period; with distance from the initial point, a significant discrepancy occurs. Based on Fig. 1(b) and Fig. 1(c), implicit schemes show a qualitative coincidence of the results. However, in the case of the moving node method, based on Fig. 1(g), which shows a comparison of the implicit scheme and Crank Nicholson with the exact solution, it follows that the result is the opposite, the purely implicit scheme by the moving node gives a good result compared to the Crank Nicholson scheme (the dotted line corresponds to the obtained purely implicit scheme, and the dotted-dotted line is obtained based on the Crank Nicholson scheme). It should also be mentioned that in the case of a non-uniform grid, the classical Crank Nicholson scheme is a first-order scheme.

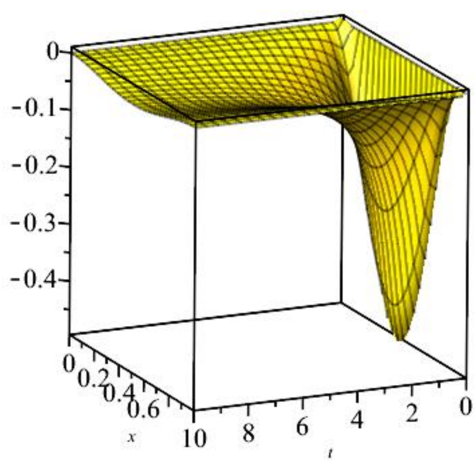
Fig. 2 shows a comparison of the mentioned scheme of the difference between the exact and approximate solutions, the difference between the exact and approximate solution is plotted vertically. Fig. 2 shows the advantage of the purely implicit scheme of the moving node.

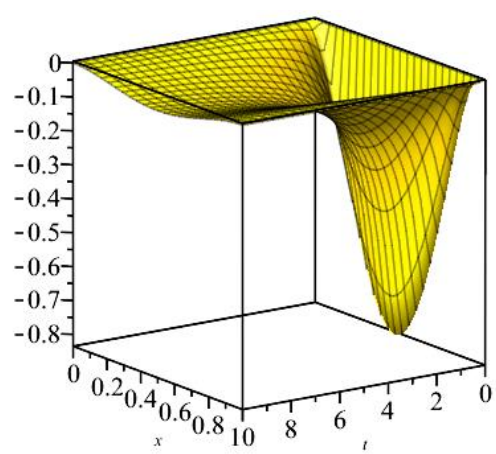
Second Problem

1. Consider equation (1) in the domain Ω = {(x,t) |0 ≤ x ≤ 1, 0 ≤ t ≤ T } with the conditions:

$$U^0(x) = x, \quad U_W(t) = 0, U_E(t) = e^{-t}, \quad f(x,t) = -x e^{-t}. \tag{10}$$

Exact solution of the problem $u(x,t) = x e^{-t}$. Fig. 3 shows a comparison of the exact and approximate solutions over the cross section $x = 0.5$ and $x = 0.2$. Solid lines correspond to the exact solution.

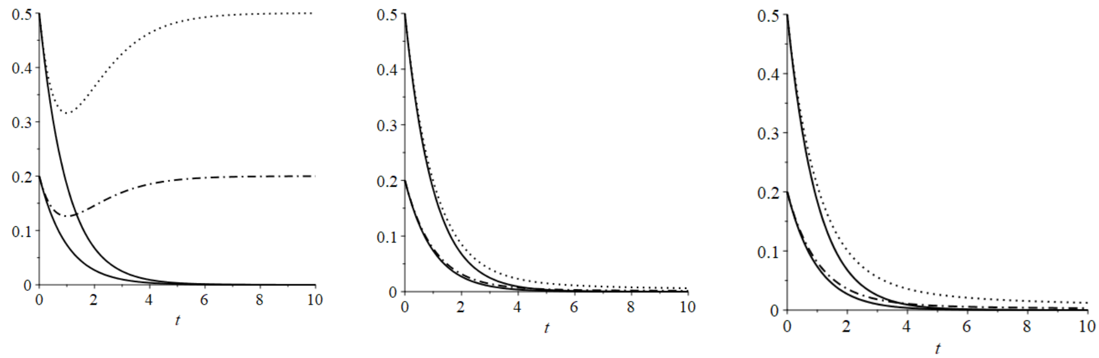




a) purely implicit scheme **(b) Crank-Nicholson scheme**

Fig. 2: Difference between exact and approximate solutions of problem 2.

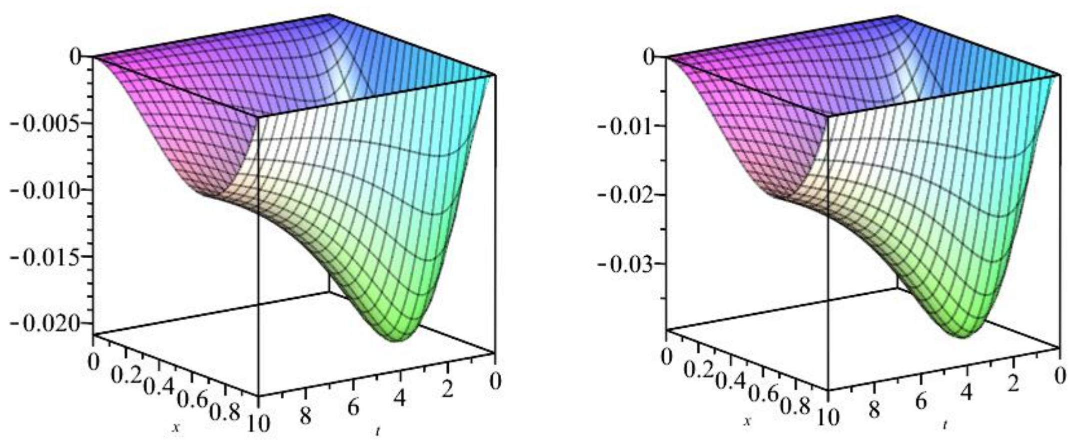
Fig. 2(a) shows a significant discrepancy between the exact and approximate solutions far from the initial point. Apparently, the beginning of the discrepancy is observed outside the stability limits of the explicit difference scheme. The implicit scheme and the Crank-Nicolson scheme (Fig. 3 (b) and (c)) give acceptable results. Fig. 4 shows the difference between the exact and approximate solutions to the problem using implicit schemes.



(a) explicit scheme (b) implicit scheme (c) Crank-Nicholson scheme

Fig. 3: Solution of equation (1) with parameters (10)

Note that the difference between the exact and approximate solutions gives a close result in form. In this example, a similar result follows as in the case of example 1: a purely implicit scheme for a moving node gives a good result compared to the Crank-Nicolson scheme (compare Fig. 3 and Fig. 4). To clarify this issue, calculations were made with a change in the Crank-Nicolson scheme: the value of the source term in formula (9) f(x,t) was replaced by f(x,t/2). In Fig. 5 shows a comparison of this change (for better clarity, the graphs corresponding to the section x=0.5 are shown in the figures). In Fig. 5, the solid line corresponds to the exact solution, the dotted line corresponds to the purely implicit scheme, the dashed line corresponds to the Crank-Nicholson scheme with the source term f(x,t/2), and the dotted-dotted line corresponds to the Crank-Nicholson scheme with the source term f(x,t).



(a) implicit scheme (c) Crank-Nicolson scheme (σ=0.5)

Fig. 4: The difference between the exact and approximate solutions to equation (1) with parameters (10)

In the initial period (0<t<4), the closest to the exact solution is the Crank-Nicholson scheme for f(x,t/2). Outside this segment, the best scheme is the purely implicit scheme.

Third Problem

Let us consider equation (1) in the domain Ω = {(x, t) |0 ≤ x ≤ 1, 0 ≤ t ≤ T} with initial and boundary conditions

$$U^0(x) = \sin \pi x + x^2, \quad U_W(t) = 0, U_E(t) = 1,$$

and with the right-hand side of the form:

$$f(x,t) = (\pi^2 - 1)\sin \pi x \cdot e^{-t} - 2.$$

Exact solution of this problem $U(x,t) = \sin \pi x e^{-t} + x^2$.

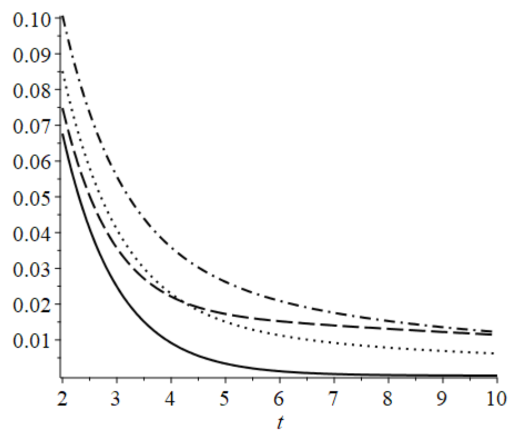
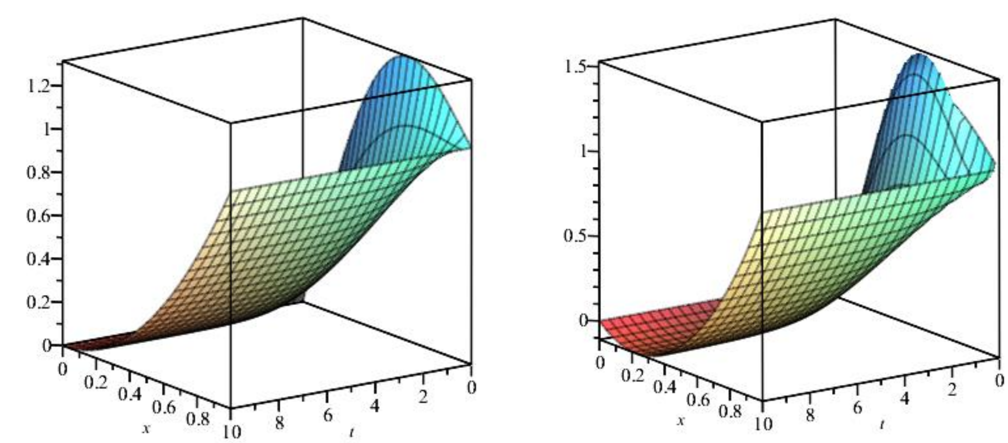
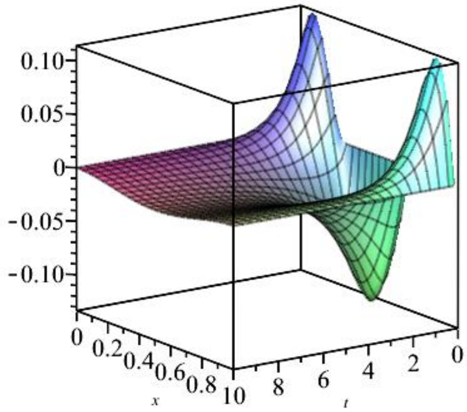


Fig. 5: Comparison of schemes of implicit schemes

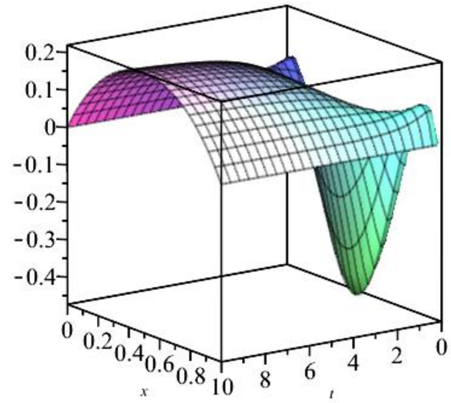
For this problem, the same tendencies are observed with respect to the comparison of schemes; i.e., the explicit scheme does not give a picture of the solution close to the exact one, the closest result occurs for the implicit scheme. Fig. 5 shows a comparison of the exact and approximate solutions.



(a) exact solution **(b) result based on the implicit scheme**



(c) the difference between the exact and approximate solved by the implicit scheme



(d) the difference between the exact and approximate solved by the Crank-Nicolson scheme

Fig. 6: Comparison of solutions

Comparison of Fig. 6 (a) and (b) visually shows the closeness of the exact and applied solutions. From the comparison of Fig. 5 c) and d) it follows that the implicit scheme is much more accurate than the Crank-Nicholson scheme.

CONCLUSION

The analysis conducted between various popular schemes on movable nodes shows that the pure implicit scheme has an advantage over the explicit scheme built on the basis of a movable node and the Crank-Nicholson scheme built on the same basis. In the case of one movable node, to obtain a rough approximate solution to the initial-boundary value problem of a parabolic equation, the purely implicit method of the movable node should be used.

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