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SERVICES FOR SCIENCE AND EDUCATION

Delay-induced Periodic Oscillations for a Three-Enterprise Interaction Model

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ABSTRACT

In the literature, two-enterprise interaction model with or without delays have been discussed by many researchers. The Hopf bifurcation in such two-enterprise model has been discussed. In this paper, we extend the two-enterprise model to a three-enterprise interaction model with delays. Time delay destroys the stability of the solutions and induces periodic oscillation for this three-enterprise model. Computer simulations are given to demonstrate the proposed results.

Keywords: three-enterprise interaction model, delay, stability, oscillation.

AMS Mathematical Subject Classification: 34K13

INTRODUCTION

In the last three decades, many researchers have considered various financial mathematical models with or without delays [1-14]. For example, Xu proposed the following without time delay model [1]:

$$\begin{cases} x_1'(t) = r_1 x_1(t)\left[1 - \frac{x_1(t)}{K_1} - \frac{\alpha(x_2(t)-c_2)^2}{K_2}\right], \\ x_2'(t) = r_2 x_2(t)\left[1 - \frac{x_2(t)}{K_2} + \frac{\beta(x_1(t)-c_1)^2}{K_1}\right], \\ x_1(0) \ge 0, x_2(0) \ge 0, \end{cases} \tag{1}$$

where variables $x_1(t)$ and $x_2(t)$ denote the output of two enterprises, respectively. r_i $(i = 1, 2)$ represent the intrinsic growth rate of the two enterprises, K_i $(i = 1, 2)$ represent the natural market carrying capacity of two enterprises under the unlimited conditions, α is the consumption coefficient of the enterprise with the output $x_2(t)$ to the one with the output $x_1(t)$, β is the transformation coefficient of the enterprise with the output $x_1(t)$ to the one with output $x_2(t)$, c_i $(i = 1, 2)$ represent the initial output of the two enterprises. The existence of periodic solutions of model (1) has been investigated by means of the coincidence degree theorem. Liao et al. have discussed the following with one delay differential system:

$$\begin{cases} x_1'(t) = r_1 x_1(t)\left[1 - \frac{x_1(t-\tau)}{K_1} - \frac{\alpha(x_2(t-\tau)-c_2)^2}{K_2}\right], \\ x_2'(t) = r_2 x_2(t)\left[1 - \frac{x_2(t-\tau)}{K_2} + \frac{\beta(x_1(t-\tau)-c_1)^2}{K_1}\right], \\ x_1(t) = \phi(t), x_2(t) = \varphi(t), t \in [-\tau, 0]. \end{cases} \tag{2}$$

By choosing the delay τ as the parameter and using the linearization method, the effect of the delay on the stability of the positive equilibrium of the model (2) and Hopf bifurcation were obtained [2]. Li et al. assumed that the interaction between two enterprises are continuous and the outputs of two enterprises satisfy a certain relation between resource and consumers, the authors proposed the following model [3]:

$$\begin{cases} x_1'(t) = r_1 x_1(t)\left[1 - \frac{x_1(t-\tau_1)}{K_1} - \frac{\alpha(x_2(t-\tau_2)-c_2)^2}{K_2}\right], \\ x_2'(t) = r_2 x_2(t)\left[1 - \frac{x_2(t-\tau_3)}{K_2} + \frac{\beta(x_1(t-\tau_4)-c_1)^2}{K_1}\right], \\ x_1(t) = \phi(t), x_2(t) = \varphi(t), t \in [-\max\{\tau_i\}, 0], \end{cases} \tag{3}$$

where τ_i ($1 \le i \le 4$) are time delays. Noting that a complete analysis regarding the distribution of roots in the complex plane of the transcendental polynomial characteristic equation with multiple different exponential terms is still a hard problem. Therefore, Li et al. assume that $\tau_1 = \tau_3 = 0$, and denote τ_2 and τ_4 by τ_1 and τ_2, namely, for the following model:

$$\begin{cases} x_1'(t) = r_1 x_1(t)\left[1 - \frac{x_1(t)}{K_1} - \frac{\alpha(x_2(t-\tau_1)-c_2)^2}{K_2}\right], \\ x_2'(t) = r_2 x_2(t)\left[1 - \frac{x_2(t)}{K_2} + \frac{\beta(x_1(t-\tau_2)-c_1)^2}{K_1}\right], \\ x_1(t) = \phi(t), x_2(t) = \varphi(t), t \in [-\max\{\tau_i\}, 0]. \end{cases} \tag{4}$$

By applying the bifurcation theory, the stability of the equilibrium point and the existence of the bifurcating periodic solutions of the model (4) were investigated.

For such time delay system several authors have studied the stability, bifurcating periodic solutions, see [4-8]. In this paper, we extend model (2) into the following three-enterprise system:

$$\begin{cases} x_1'(t) = r_1 x_1(t)\left[1 - \frac{x_1(t-\tau_1)}{K_1} - \frac{x_2(t-\tau_2)}{K_2} - \frac{\alpha(x_3(t-\tau_3)-c_3)^2}{K_3}\right], \\ x_2'(t) = r_2 x_2(t)\left[1 - \frac{x_2(t-\tau_3)}{K_2} - \frac{x_3(t-\tau_3)}{K_3} - \frac{\beta(x_1(t-\tau_4)-c_1)^2}{K_1}\right], \\ x_3'(t) = r_3 x_3(t)\left[1 - \frac{x_3(t-\tau_3)}{K_3} - \frac{x_1(t-\tau_1)}{K_1} + \frac{\gamma(x_2(t-\tau_2)-c_2)^2}{K_2}\right] \\ x_1(t) = \varphi_1(t), x_2(t) = \varphi_2(t), x_3(t) = \varphi_3(t), t \in [-\max\{\tau_i\}, 0], \end{cases} \tag{5}$$

where $\tau_i \ge 0 (1 \le i \le 3)$, variables $x_i(t)$ (i = 1, 2, 3) denote the output of three enterprises, respectively, r_i (i = 1, 2, 3) represent respectively the intrinsic growth rate of three enterprises, K_i (i = 1, 2, 3) are the natural market carrying capacity of three enterprises under the unlimited conditions; α is the consumption coefficient of the enterprise with the output $x_3(t)$ to the output $x_1(t)$ and $x_2(t)$, β is the consumption coefficient of the enterprise with the output $x_1(t)$ to the output $x_2(t)$ and $x_3(t)$, and γ denotes the transformation coefficient of the enterprise with the output $x_2(t)$ to the output $x_3(t)$ and $x_1(t)$; c_i (i = 1, 2, 3) denote the initial output of three enterprises. All parameters are positive real numbers.

Time delay induced oscillation is a very interesting phenomenon. Belair et al studied the stability and delay induced oscillations in a neural network model [15]. Bodnar and Bartlomiejczyk have discussed the stability of delay induced oscillation in gene expression model [16]. Tabbert et al. investigated the oscillatory motion of dissipative solitons induced by delay-feedback in inhomogeneous Kerr resonators [17]. Xu considered the stability and delay induced oscillation in a competitor-competitor-mutualist Lotka-Volterra model [18]. Xu et al. also considered delay induced periodic oscillation for fractional-order neural networks [19]. Bratsun et al. discussed delay induced oscillation in a thermal convection loop under negative feedback control with noise [20]. In this paper, by applying the mathematical analysis method, the periodic oscillation induced by time delays of the model (5) is investigated.

System (5) can be rewritten as the following form:

$$\begin{cases} y_1'(t) = (y_1(t)+c_1)[d_1 - a_1 y_1(t-\tau_1) - b_2 y_2(t-\tau_2) - k_3 y_3^2(t-\tau_3)], \\ y_2'(t) = (y_2(t)+c_2)[d_2 - a_2 y_2(t-\tau_2) - b_3 y_3(t-\tau_3) - k_1 y_1^2(t-\tau_1)], \\ y_3'(t) = (y_3(t)+c_3)[d_3 - a_3 y_3(t-\tau_3) - b_1 y_1(t-\tau_1) + k_2 y_2^2(t-\tau_2)], \end{cases} \tag{6}$$

where

$$y_i(t) = x_i(t) - c_i, a_i = \frac{r_i}{K_i}\ (i=1,2,3), b_1 = \frac{r_3}{K_1}, b_2 = \frac{r_1}{K_2}, b_3 = \frac{r_2}{K_3}, k_1 = \frac{\alpha r_1}{K_1}, k_2 = \frac{\beta r_2}{K_2}, k_3 = \frac{\gamma r_3}{K_3},\ d_1 = r_1 - a_1 c_1 - b_2 c_2,\ d_2 = r_2 - a_2 c_2 - b_3 c_3, d_3 = r_3 - a_3 c_3 - b_1 c_1.$$

PRELIMINARIES

Obviously, $(-c_1, -c_2, -c_3)$ is an equilibrium point of system (6). However, we are concerned about the positive equilibrium point. Assume that (y_1^*, y_2^*, y_3^*) is a positive equilibrium point, from (6) we have

$$\begin{cases} d_1 - a_1 y_1^* - b_2 y_2^* - k_3 y_3^{*2} = 0, \\ d_2 - a_2 y_2^* - b_3 y_3^* - k_1 y_1^{*2} = 0, \\ d_3 - a_3 y_3^* - b_1 y_1^* + k_2 y_2^{*2} = 0, \end{cases} \tag{7}$$

Then make the change of variables as $z_1(t) \to y_1(t) - y_1^*, z_2(t) \to y_2(t) - y_2^*,\ z_3(t) \to y_3(t) - y_3^*$, from (6) we have

$$\begin{cases} z_1'(t) = p_{11} z_1(t-\tau_1) + p_{12} z_2(t-\tau_2) + p_{13} z_3(t-\tau_3) \\ +z_1(t)[-a_1 z_1(t-\tau_1) - b_2\, z_2(t-\tau_2) - 2k_3\, y_3^* z_3(t-\tau_3) - k_3 z_3^2(t-\tau_3)], \\ z_2'(t) = p_{21} z_1(t-\tau_1) + p_{22} z_2(t-\tau_2) + p_{23} z_3(t-\tau_3) \\ +z_2(t)[-a_2 z_2(t-\tau_2) - b_3\, z_3(t-\tau_3) - 2k_1\, y_1^* z_1(t-\tau_1) - k_1 z_1^2(t-\tau_1)], \\ z_3'(t) = p_{31} z_1(t-\tau_1) + p_{32} z_2(t-\tau_2) + p_{33} z_3(t-\tau_3) \\ +z_3(t)[-a_3 z_3(t-\tau_3) - b_1\, z_1(t-\tau_1) + 2k_2\, y_2^* z_2(t-\tau_2) + k_2 z_2^2(t-\tau_2)], \end{cases} \tag{8}$$

where

$$p_{11} = -a_1(y_1^*+c_1), p_{12} = -b_2(y_1^*+c_1), p_{13} = -2k_3\, y_3^*(y_1^*+c_1), p_{21} = -2k_1\, y_1^*(y_2^*+c_2), p_{22} = -a_2(y_2^*+c_2), p_{23} = -b_3(y_2^*+c_2), p_{31} = -b_1(y_3^*+c_3),\ p_{32} = 2k_2\, y_2^*(y_3^*+c_3), p_{33} = -a_3(y_3^*+c_3).$$

Thus, system (8) can be written as a matrix form:

$$z'(t) = Pz(t-\tau) + G(z(t), z(t-\tau)) \tag{9}$$

where $z(t) = [z_1(t), z_2(t), z_3(t)]^T, z(t-\tau) = [z_1(t-\tau_1), z_2(t-\tau_2), z_3(t-\tau_3)]^T$, matrix

$$P = (p_{ij})_{3\times 3} = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$

$$G\big(z(t), z(t-\tau)\big) = (z_1(t)[-a_1 z_1(t-\tau_1) - b_2\, z_2(t-\tau_2) - 2k_3\, y_3^* z_3(t-\tau_3) - k_3 z_3^2(t-\tau_3)], \cdots, z_3(t)[-a_3 z_3(t-\tau_3) - b_1\, z_1(t-\tau_1) + 2k_2\, y_2^* z_2(t-\tau_2) + k_2 z_2^2(t-\tau_2)]\,)^T.$$

The linearized system of (9) is the following:

$$z'(t) = Pz(t-\tau) \tag{10}$$

Obviously, the zero equilibrium point of system (9) corresponds to the positive equilibrium point (y_1^*, y_2^*, y_3^*) of system (6). Therefore, in order to discuss the stability or instability of the positive equilibrium point (y_1^*, y_2^*, y_3^*) of the system (6), we only need deal with the stability or instability of the zero equilibrium point of system (8) or (9).

Definition 1

Assume that the solutions of system (6) are convergent when time delays $\tau_i < \tau_0 (i = 1, 2, 3)$, and there is an oscillatory solution as $\tau_i > \tau_0 (i = 1, 2, 3)$, which is called time delay-induced oscillation.

Lemma 1

All solutions of system (5) (or (6)) are bounded.

Proof

It is known that time delay affects the stability of the solutions, it cannot affect the boundedness of the solutions [21]. Therefore, we can only consider the boundedness of the following without time delay system:

$$\begin{cases} x_1'(t) = r_1 x_1(t)\left[1 - \frac{x_1(t)}{K_1} - \frac{x_2(t)}{K_2} - \frac{\alpha(x_3(t)-c_3)^2}{K_3}\right], \\ x_2'(t) = r_2 x_2(t)\left[1 - \frac{x_2(t)}{K_2} - \frac{x_3(t)}{K_3} - \frac{\beta(x_1(t)-c_1)^2}{K_1}\right], \\ x_3'(t) = r_3 x_3(t)\left[1 - \frac{x_3(t)}{K_3} - \frac{x_1(t)}{K_1} + \frac{\gamma(x_2(t)-c_2)^2}{K_2}\right]. \end{cases} \tag{11}$$

Noting that all parameters are positive real numbers in system (5). Since we only concern the boundedness of positive solutions, from the first equation of the system (11) we have

$$x_1'(t) \le r_1 x_1(t)\left[1 - \frac{x_1(t)}{K_1}\right] = r_1 x_1(t) - \frac{r_1}{K_1} x_1^2\,(t) \tag{12}$$

Equation (12) is a Bernoulli' equation of $n = 2$. So we have $x_1(t) < K_1$. The same as for the second equation of system (11) we get $x_2(t) < K_2$. Now for the third equation of the system (11) we have

$$x_3'(t) \le r_3 x_3(t)\left[1 - \frac{x_3(t)}{K_3} + \frac{\gamma (K_2)^2}{K_2}\right] = r_3 x_3(t)\left[1 - \frac{r_3}{K_3} x_3^2\,(t) + \gamma K_2\right] \tag{13}$$

From equation (13) we have $x_3(t) < \frac{r_3 K_3 + K_3\sqrt{r_3^2 + 4\gamma K_2 K_3}}{2r_3}$. Thus, all solutions of the system (5) are positively bounded. The proof is completed.

Lemma 2

Assume that P is a nonsingular matrix, then system (8) has a unique zero equilibrium point, implying that system (6) has a unique positive equilibrium point (y_1^*, y_2^*, y_3^*).

Proof

Consider system (10), since P is a nonsingular matrix, then the system (10) has a unique zero equilibrium point according to the basic algebraic theory, implying that system (8) has a unique zero equilibrium point. Therefore, system (6) has a unique positive equilibrium point (y_1^*, y_2^*, y_3^*).

DELAY-INDUCED PERIODIC OSCILLATION AROUND THE POSITIVE EQUILIBRIUM POINT

Theorem 1

Let the three eigenvalues of the matrix P be ρ_1, ρ_2, ρ_3, and $|\rho_*| = \min\{|\rho_1|, |\rho_2|, |\rho_3|\}, \tau = \min\{\tau_1, \tau_2, \tau_3\}$. Suppose that Lemma 2 holds for selected parameter values. Assume that the following inequality is satisfied:

$$|\rho_*| e\,\tau > 1 \tag{14}$$

Then there exists delay induced periodic oscillation.

Proof

Since the parameters in system (5) are positive real numbers, and (y_1^*, y_2^*, y_3^*).is the unique positive equilibrium point, it is easy to know that all characteristic values of matrix P are negative or have negative real part of complex numbers. Therefore, the trivial solution of system (10) is stable if delay is zero. Naturally, the trivial solution of system (10) is still stable for small delays, noting that $G(z(t), z(t-\tau))$ is a higher order infinitesimal as $z_i(t) \to 0$ $(i = 1, 2, 3)$. Therefore, the stability of trivial solution of system (10) implies the stability of the trivial solution of system (9). This means that the unique positive equilibrium point

(y_1^*, y_2^*, y_3^*). of system (6) is stable when time delays are small. On the other hand, the characteristic equation of system (10) is the following:

$$\det\left(\lambda I_{ij} - p_{ij} e^{-\lambda \tau_j}\right) = 0, \tag{15}$$

where I_{ij} is the 3×3 identity matrix. The equivalent form of the system (15) is that:

$$\prod_{j=1}^{3}\left(\lambda - \rho_j e^{-\lambda \tau_j}\right) = 0 \tag{16}$$

Thus, we are led to an investigation of the nature of the roots, for simply, let $j = 1$, we have

$$\lambda - \rho_1 e^{-\lambda \tau_1} = 0 \tag{17}$$

(17) is a transcendental equation. There are infinitely many solutions for such a transcendental equation. It may have a positive real part eigenvalue. Indeed, Let $\lambda = \lambda_1 + i\lambda_2, \rho_1 = \rho_{11} + i\rho_{12}$, where $\lambda_1 = Re(\lambda), \rho_{11} = Re(\rho_1)$, $\lambda_2 = Im(\lambda), \rho_{12} = Im(\rho_1)$. If $\rho_{12} = 0$, means that is a negative real number. Thus, we have $\lambda_1 = e^{-\lambda_1 \tau_1}(\rho_{11}\cos(\lambda_2 \tau_1) + \rho_{12}\sin(\lambda_2 \tau_1))$. Obviously, if $\rho_{11}\cos(\lambda_2 \tau_1) + \rho_{12}\sin(\lambda_2 \tau_1) > 0$, then $\lambda_1 > 0$. In other words, we cannot guarantee that the transcendental equation (17) always has negative eigenvalue. When delay increases to a critical value, periodic oscillation of the solutions occurs.

In order to find the critical value of time delay we assume that $\lambda < 0$ in (17). From (17) using formula $e^x > ex$ we have

$$1 = |\rho_1| \frac{e^{-\lambda \tau_1}}{|\lambda|} = |\rho_1| \frac{\tau_1 e^{|\lambda| \tau_1}}{\tau_1 |\lambda|} \geq |\rho_1| \tau_1 e \geq |\rho_*| \tau e \tag{18}$$

The last inequality of (18) contradicts condition $|\rho_*|\tau e > 1$. This means that $\lambda < 0$ does not hold in equation (17). Therefore, the trivial solution of the system (9) is unstable under restrictive condition $|\rho_*|\tau e > 1$, implying that the trivial solution of system (8) is unstable. This means that the positive equilibrium point (y_1^*, y_2^*, y_3^*) of system (6) is unstable. In this case, condition (14) implies that delay induces periodic oscillation occurs.

Theorem 2

Suppose that lemma 2 holds for selected parameter values. Assume that the following inequality for some $i \in \{1, 2, 3\}$ holds:

$$|p_{ii}| \tau_i e > 1 + e \sum_{j=1, j \neq i}^{3} |p_{ij}| \tau_j \tag{19}$$

Then there exists delay-induced periodic oscillation.

Proof

The characteristic equation corresponding to system (9) is (15). Suppose that the trivial solution of system (9) is stable. Then system (15) must have a real negative root say λ_0 such that

$$\det\left(\lambda_0 I_{ij} - p_{ij} e^{-\lambda_0 \tau_j}\right) = 0. \tag{20}$$

According to the very known Gershgorin's circle theorem [22], for some $i \in \{1, 2, 3\}$ we have

$$|\lambda_0 - p_{ii} e^{-\lambda_0 \tau_i}| \leq \sum_{j=1, j\neq i}^{3} |p_{ij}|\, e^{-\lambda_0 \tau_j} \tag{21}$$

From (21) and

$$\begin{aligned} |\lambda_0| &= |p_{ii} e^{-\lambda_0 \tau_i} + \lambda_0 - p_{ii} e^{-\lambda_0 \tau_i}| \\ &\geq |p_{ii}| e^{-\lambda_0 \tau_i} - |\lambda_0 - p_{ii} e^{-\lambda_0 \tau_i}| \\ &\geq |p_{ii}| e^{-\lambda_0 \tau_i} - \sum_{j=1, j\neq i}^{3} |p_{ij}|\, e^{-\lambda_0 \tau_j} \end{aligned} \tag{22}$$

Namely,

$$|\lambda_0| + \sum_{j=1, j\neq i}^{3} |p_{ij}|\, e^{-\lambda_0 \tau_j} \geq |p_{ii}| e^{-\lambda_0 \tau_i} \tag{23}$$

Thus from (23) we have

$$1 + \sum_{j=1, j\neq i}^{3} \tau_j |p_{ij}| \frac{e^{-\lambda_0 \tau_j}}{|\lambda_0| \tau_j} \geq |p_{ii}| \frac{e^{-\lambda_0 \tau_i}}{|\lambda_0| \tau_i} \tag{24}$$

Again, using formula $e^x > ex$ we have

$$1 + e \sum_{j=1, j\neq i}^{3} |p_{ij}|\, \tau_j \geq \ |p_{ii}| \tau_i e \tag{25}$$

But (25) contradicts (19). Therefore, the trivial solution of system (9) is unstable, implying that the positive equilibrium point (y_1^*, y_2^*, y_3^*) of the system (5) is unstable. Thus, delay-induced periodic oscillation occurs. The proof is completed.

SIMULATION RESULTS

This simulation is based on system (6). We first select $a_1 = 1.92, a_2 = 1.85, a_3 = 1.88, b_1 = 0.25, b_2 = 0.26, b_3 = 0.22, c_1 = 0.18, c_2 = 0.15, c_3 = 0.12, d_1 = 0.32, d_2 = 0.28, d_3 = 0.35,$ $k_1 = 0.55, k_2 = 0.42, k_3 = 0.68$. Then we have

$$P_1 = \begin{pmatrix} -0.6280 & -0.0851 & -0.0746 \\ -0.0357 & -0.4087 & -0.0486 \\ -0.0719 & -0.0176 & -0.4309 \end{pmatrix}$$

The eigenvalues of P_1 are $\rho_1 = -0.6779$, $\rho_2 = -0.4645, \rho_3 = -0.4352$. The unique positive equilibrium point is $(y_1^*, y_2^*, y_3^*) = (0.1445, 0.1273, 0.1478)$. The equilibrium point is stable when time delays are selected as 1.45, 1.48, 1.46, and1.75, 1.78, 1.76, respectively, see Fig. 1; also when time delays are selected as 2.05, 2.08, 2.06, and 2.25, 2.28, 2.26, respectively, the equilibrium point is still stable (see Fig. 2). When time delays are selected as $\tau_1 = 2.45, \tau_2 = 2.48, \tau_3 = 2.46$, and $\tau_1 = 2.95, \tau_2 = 2.98, \tau_3 = 2.96$, respectively, we have $\tau = 2.45$ and $|\rho_*| = 0.4352$. Therefore, $|\rho_*| e\, \tau = 0.4352 \times 2.45 \times 2.7183 = 2.8983 > 1$, based on Theorem 1, delay-induced periodic oscillation occurs (see Fig. 3). Then we select $a_1 = 1.75, a_2 = 1.65$, $a_3 = 1.55, b_1 = 0.45, b_2 = 0.46$, $b_3 = 0.42, c_1 = 0.28$, $c_2 = 0.25, c_3 = 0.22, d_1 = 0.52, d_2 = 0.48, d_3 = 0.50, k_1 = 0.65, k_2 = 0.62, k_3 = 0.78$. Thus we have

$$P_2 = \begin{pmatrix} -0.8477 & -0.2172 & -0.2085 \\ -0.1213 & -0.7590 & -0.1392 \\ -0.2243 & -0.1293 & -0.7727 \end{pmatrix}$$

The eigenvalues of P_2 are $\rho_1 = -1.1630, \rho_2 = -0.6082 + 0.0476i, \rho_3 = -0.6082 - 0.0476i$
The unique positive equilibrium point is $(y_1^*, y_2^*, y_3^*) = (0.2027, 0.2092, 0.2785)$.

The equilibrium point $(0.2027, 0.2092, 0.2785)$ is stable when time delays are selected as $1.02, 0.97, 1.00$, and $1.12, 1.07, 1.10$, respectively (see Fig. 4); when time delays are selected as $1.32, 1.27, 1.30$, and $1.42, 1.37, 1.40$, respectively, the equilibrium point is still stable (see Fig. 5). When time delays are selected as $\tau_1 = 1.52, \tau_2 = 1.47, \tau_3 = 1.50$, and $\tau_1 = 1.60, \tau_2 = 1.55, \tau_3 = 1.58$, respectively, we have $|p_{11}|\tau_1 e$ = 0.8477 × 1.52 × 2.7183 = 3.5025, and $1 + e \times (|p_{12}|\tau_2 + |p_{13}|\tau_3) = 1 + 2.7183 \times (0.2172 \times 1.47 + 0.2085 \times 1.50) = 2.7352$. Therefore, $|p_{11}|\tau_1 e > 1 + e \times (|p_{12}|\tau_2 + |p_{13}|\tau_3)$ holds. Based on theorem 2, delay-induced periodic oscillation occurs (see Fig. 6).

CONCLUSION

In this paper, we have discussed a three-enterprise interaction model with three different delays. The existence of periodic oscillatory solution which is induced by delays has been proposed. Delay-induced periodic oscillation is a very interesting problem. According to the computer simulation, the oscillatory frequencies are the same when delay induces periodic oscillation. However, the oscillatory amplitudes are different from each other. It is pointed out that the present result is only a sufficient condition.

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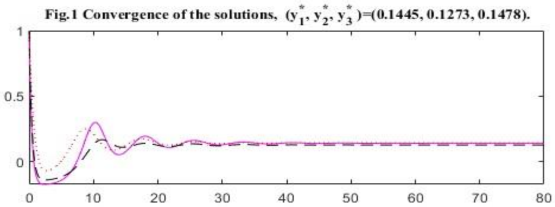
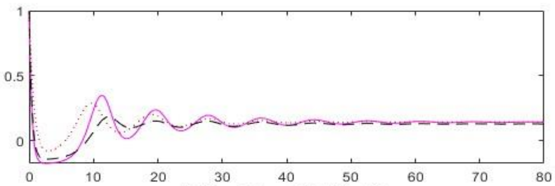
Fig.1 Convergence of the solutions, (y_1^*, y_2^*, y_3^*)=(0.1445, 0.1273, 0.1478).

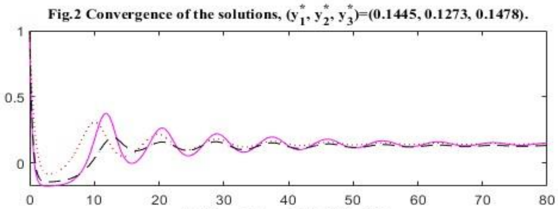


(a) Time delays: 1.75, 1.78, 1.76.

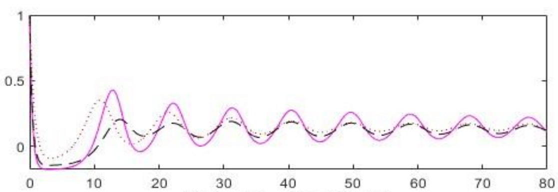


(b) Time delays: 1.95, 1.98, 1.96.

Fig.2 Convergence of the solutions, (y_1^*, y_2^*, y_3^*)=(0.1445, 0.1273, 0.1478).

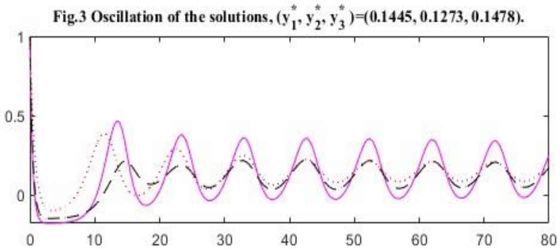


(a) Time delays: 2.05, 2.08, 2.06.

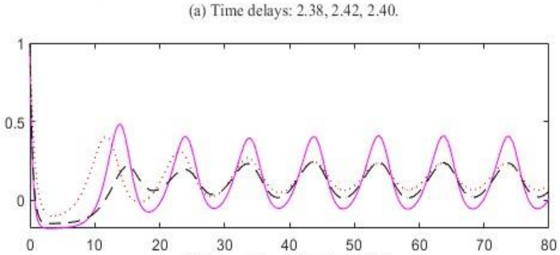


(b) Time delays: 2.25, 2.28, 2.26.

Fig.3 Oscillation of the solutions, (y_1^*, y_2^*, y_3^*)=(0.1445, 0.1273, 0.1478).

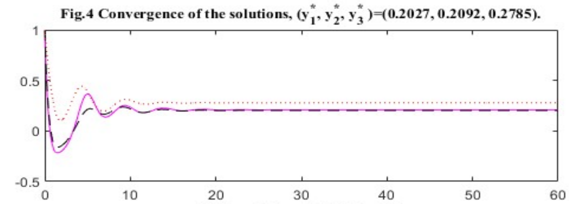


(a) Time delays: 2.38, 2.42, 2.40.

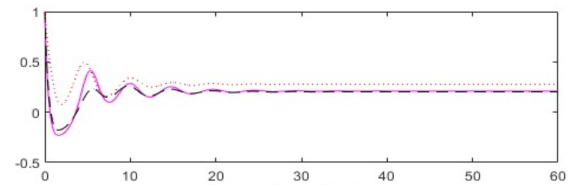


(b) Time delays: 2.45, 2.48, 2.46.

Fig.4 Convergence of the solutions, (y_1^*, y_2^*, y_3^*)=(0.2027, 0.2092, 0.2785).

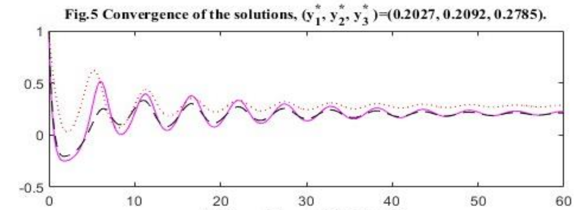


(a) Time delays: 1.02, 0.97, 1.00.

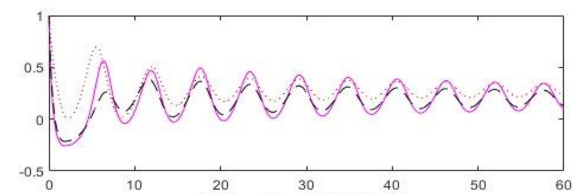


(b) Time delays: 1.12, 1.07, 1.10.

Fig.5 Convergence of the solutions, (y_1^*, y_2^*, y_3^*)=(0.2027, 0.2092, 0.2785).

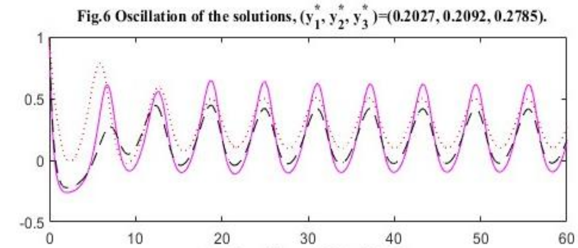


(a) Time delays: 1.32, 1.27, 1.30.

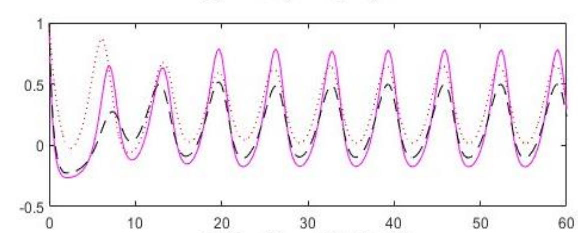


(b) Time delays: 1.42, 1.37, 1.40.

Fig.6 Oscillation of the solutions, (y_1^*, y_2^*, y_3^*)=(0.2027, 0.2092, 0.2785).



(a) Time delays: 1.52, 1.47, 1.50.



(b) Time delays: 1.60, 1.55, 1.58.