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# Beyond g>r: Transitional Dynamics of the Debt-to-GDP Ratio

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**Abstract:** The present paper analyzes the transitional dynamics of the government-debt-to-GDP ratio in an economy with constant interest rates, economic growth, and fiscal deficit growth. By explicitly modeling the joint evolution of government debt, fiscal deficits, and GDP, the present paper derives closed-form expressions for the debt-to-GDP ratio and characterizes its dynamic behavior. The analysis shows that even when standard long-run sustainability conditions—such as an interest rate lower than the economic growth rate—are satisfied, the debt-to-GDP ratio need not evolve monotonically. Instead, it may exhibit a single-peaked trajectory, rising temporarily before converging in the long run. The present paper identifies precise conditions under which such non-monotonic dynamics arise and illustrates them with numerical examples.

**Keywords:** transitional dynamics, single-peaked trajectory, interest-growth differential, growth rate of fiscal deficit, government-debt-to-GDP ratio.

## INTRODUCTION

Sustainability of public debt has long been analyzed through the lens of interest-growth differential. The seminal contribution of **Evsey D. Domar** (1944) demonstrates that, under a constant primary deficit relative to national income, the government-debt-to-GDP ratio remains bounded as long as the growth rate of the economy exceeds the interest rate ($r < g$). This insight, often referred to as *Domar's condition*, has become a cornerstone of modern analyses of public debt sustainability.

Building on this framework, a large body of subsequent literature has emphasized that debt dynamics are jointly determined by the interest-growth differential and the primary balance. In particular, recent influential contributions have renewed interest in the case where interest rates are persistently below growth rates. For example, **Olivier Blanchard** (2019) argues that when $r < g$ holds over long periods, the fiscal costs of public debt may be substantially smaller than traditionally assumed. Related discussions, such as **Jason Furman** (2020) and **Paolo Mauro** and **Jing Zhou** (2021), further examine whether a negative interest-growth differential can relax conventional fiscal constraints and justify higher levels of public debt.

While this literature has significantly reshaped the policy debate on fiscal sustainability, it has largely focused on long-run averages and steady-state properties of debt dynamics. In most analyses, the primary balance is assumed to be stationary relative to GDP, implying that when $r < g$ holds, the government-debt-to-GDP ratio converges monotonically to a finite steady-state level—even in the presence of a permanent primary deficit. As a result, relatively little attention has been paid to the *transitional dynamics* of public debt along the adjustment path.

However, fiscal policy is typically evaluated over finite horizons, during which transitional dynamics may play a crucial role. Even when long-run convergence is guaranteed, the debt-to-GDP ratio need not evolve monotonically. Instead, it may initially rise, reach a peak, and only subsequently decline. Such non-monotonic behavior can be misinterpreted as evidence of fiscal unsustainability if attention is restricted to short- or medium-run outcomes. Earlier studies on debt dynamics, such as **Henrique Bohn** (1998), highlight the importance of dynamic considerations, but do not explicitly characterize the conditions under which transitional debt paths may exhibit temporary deterioration despite long-run sustainability.

The present paper complements and extends the existing literature by explicitly analyzing the transitional dynamics of the government-debt-to-GDP ratio in an economy where the growth rate of fiscal deficits may differ from the growth rate of output. By jointly modeling the evolution of government debt, fiscal deficits, and GDP, the analysis derives closed-form expressions for the debt-to-GDP ratio and identifies the conditions under which it converges, diverges, or follows a non-monotonic path. In particular, the paper shows that even when the standard long-run sustainability conditions—such as r < g and a sufficiently low growth rate of fiscal deficits—are satisfied, the debt-to-GDP ratio can exhibit a single-peaked trajectory, increasing for an initial period before eventually declining.

By highlighting the importance of transitional dynamics, this paper qualifies common interpretations of the r < g condition as a sufficient safeguard against fiscal risk. The results emphasize the need to distinguish carefully between short-run debt dynamics and long-run fiscal solvency when evaluating fiscal performance.

The remainder of the paper is organized as follows. Section 2 presents the analytical framework. Sections 3 and 4 derive the dynamics of the government-debt-to-GDP ratio under alternative relationships between the interest rate and the growth rate of fiscal deficits. Section 5 concludes.

## BASIC MODEL

We consider an economy in which interest rate, growth rate of fiscal deficit, and economic growth rate are constant over time and are denoted by $r$, $d$, and $g$, respectively.

If we let $S(t)$, $D(t)$, and $Y(t)$denote government debt outstanding, the fiscal deficit, and GDP in period $t$, laws of motion for the government debt, the fiscal deficit, and GDP are given by

$$S(t) = (1 + r)S(t - 1) + D(t - 1),$$

$$D(t) = (1 + d)D(t - 1),$$

$$Y(t) = (1 + g)Y(t - 1),$$

Let the initial values of $S(t)$, $D(t)$, and $Y(t)$be denoted by $S_0$, $D_0$, and $Y_0$, respectively, then,

$$S(t) = (1 + r)S(t - 1) + D(t - 1) = (1 + r)S(t - 1) + (1 + d)^{t-1}D_0.$$

Hence, we have

$$S(t) = (1+r)^t S_0 + \sum_{k=0}^{t-1} (1+r)^{t-1-k} (1+d)^k D_0.$$

Assuming $r \neq d$, this expression can be written as

$$S(t) = (1+r)^t S_0 + \frac{(1+r)^t - (1+d)^t}{r-d} D_0.$$

Therefore, the general expression for the government-debt-to-GDP ratio is given by

$$\frac{S(t)}{Y(t)} = \frac{S_0}{Y_0}\left(\frac{1+r}{1+g}\right)^t + \frac{D_0}{Y_0}\frac{1}{r-d}\left[\left(\frac{1+r}{1+g}\right)^t - \left(\frac{1+d}{1+g}\right)^t\right]. \quad (1)$$

It follows immediately from (1) that when $g > r$ and $g > d$, the government-debt-to-GDP ratio does not diverge.

Next, differentiating equation (1) with respect to $t$ and setting the derivative equal to zero yields

$$\left(\frac{S_0}{Y_0} + \frac{D_0}{Y_0}\frac{1}{r-d}\right)\left(\frac{1+r}{1+g}\right)^t \ln\left(\frac{1+r}{1+g}\right) - \frac{D_0}{Y_0}\frac{1}{r-d}\left(\frac{1+d}{1+g}\right)^t \ln\left(\frac{1+d}{1+g}\right) = 0.$$

Rearranging this condition gives

$$\left(\frac{1+r}{1+d}\right)^t = \frac{\frac{D_0}{Y_0}\frac{1}{r-d}\ln\left(\frac{1+d}{1+g}\right)}{\left(\frac{S_0}{Y_0} + \frac{D_0}{Y_0}\frac{1}{r-d}\right)\ln\left(\frac{1+r}{1+g}\right)}.$$

Taking logarithms of both sides and rearranging, the value of $t$that maximizes $S(t)/Y(t)$is given by

$$t^* = \frac{\ln\left(\frac{D_0 \ln\left(\frac{1+d}{1+g}\right)}{(S_0(r-d)+D_0)\ln\left(\frac{1+r}{1+g}\right)}\right)}{\ln\left(\frac{1+r}{1+d}\right)}. \quad (2)$$

## DYNAMICS OF THE GOVERNMENT-DEBT-TO-GDP RATIO WHEN $r \neq d$

Based on the above discussion, this section derives the dynamics of the government-debt-to-GDP Ratio. When $r > d$, the denominator of (2) is positive. Hence, if

$$\frac{D_0 \ln\left(\frac{1+d}{1+g}\right)}{(S_0(r-d)+D_0)\ln\left(\frac{1+r}{1+g}\right)} > 1, \quad (3)$$

then $t^* > 0$, and the government-debt-to-GDP ratio follows a single-peaked path. By contrast, if

$$\frac{D_0 \ln\left(\frac{1+d}{1+g}\right)}{(S_0(r-d)+D_0)\ln\left(\frac{1+r}{1+g}\right)} \leq 1, \quad (4)$$

then $t^* < 0$, and $S(t)/Y(t)$is monotonically decreasing.

By using the approximation $\log(1+x) \approx x$, condition (3), which can be rewritten as

$$D_0 \ln\left(\frac{1+d}{1+r}\right) > S_0(r-d)\ln\left(\frac{1+r}{1+g}\right),$$

becomes

$$D_0(d-r) > S_0(r-d)(r-g),$$

which can be simplified to

$$\frac{D_0}{S_0} < g - r. \quad (5)$$

On the other hand, when $r < d$, the denominator of (2) is negative. In this case, if

$$\frac{D_0 \ln\left(\frac{1+d}{1+g}\right)}{(S_0(r-d)+D_0)\ln\left(\frac{1+r}{1+g}\right)} < 1, \quad (6)$$

then $t^* > 0$ and $S(t)/Y(t)$ is single-peaked, whereas if

$$\frac{D_0 \ln\left(\frac{1+d}{1+g}\right)}{(S_0(r-d)+D_0)\ln\left(\frac{1+r}{1+g}\right)} \geq 1, \quad (7)$$

then $t^* < 0$ and the government-debt-to-GDP ratio is monotonically decreasing. Proceeding in the same manner as in the case $r > d$, condition (6) can be simplified to

$$\frac{D_0}{S_0} > g - r. \quad (8)$$

The above discussion can be summarized in the following proposition.

**Proposition 1**

Suppose that $g > r$ and $g > d$. If either (1) $r > d$and $\frac{D_0}{S_0} < g - r$, or (2) $r < d$and $\frac{D_0}{S_0} > g - r$, then the government-debt-to-GDP ratio $S(t)/Y(t)$ converges to zero in the steady state, while increasing for an initial period.

**Numerical Example 1(1)**

When $r = 0.02$, $g = 0.08$, $d = 0.005$, $S_0/Y_0 = 0.30$, and $D_0/Y_0 = 0.30$ (that is, the case where $r > d$ and $D_0/S_0 < g - r$), the time path of the government-debt-to-GDP ratio is given as follows:

| $t$ | **0** | **5** | **10** | **14** | **15** | **16** | **20** |
|---|---|---|---|---|---|---|---|
| $S(t)/Y(t)$ | 0.3000 | 1.2984 | 1.7244 | 1.8177 | 1.8181 | 1.8115 | 1.7307 |

In this case, $S(t)/Y(t)$ increases initially, reaches a maximum at $t = 15$, and then declines thereafter. Accordingly, the government-debt-to-GDP ratio follows a single-peaked path, as illustrated in Figure 1.

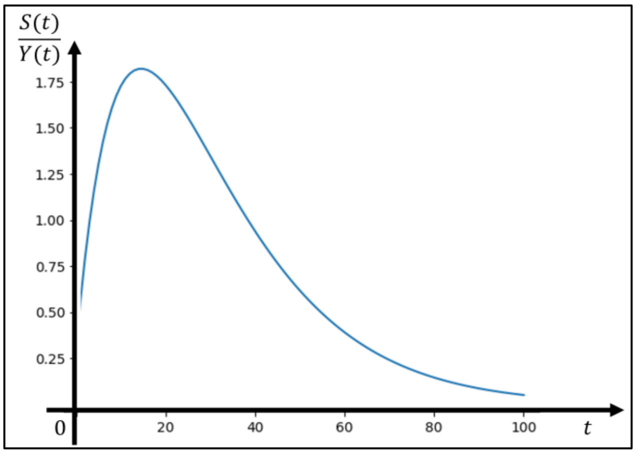


**Figure 1: A single-peaked path of the government-debt-to-GDP ratio ($r > d$ case).**

This numerical example corresponds to case (1) of Proposition 1 and illustrates that when $r > d$, the government-debt-to-GDP ratio can exhibit a temporary increase before converging in the long run, provided that the initial deficit-to-debt ratio is sufficiently small.

**Numerical Example 1(2)**

When $r = 0.02$, $d = 0.05$, $g = 0.10$, $S_0/Y_0 = 0.3$, and $\frac{D_0}{Y_0} = 0.3$ (that is, the case where $r < d$ and $D_0/S_0 > g - r$), the time path of the government-debt-to-GDP ratio is given by:

| $t$ | 0 | 5 | 10 | 15 | 16 | 17 | 20 |
|---|---|---|---|---|---|---|---|
| $S(t)/Y(t)$ | 0.3000 | 1.2749 | 1.7213 | 1.8515 | 1.8526 | 1.8474 | 1.8015 |

In this case, $S(t)/Y(t)$increases in the initial periods, reaches its maximum at $t = 16$, and then declines thereafter. Accordingly, the government-debt-to-GDP ratio follows a single-peaked path, as illustrated in Figure 2.

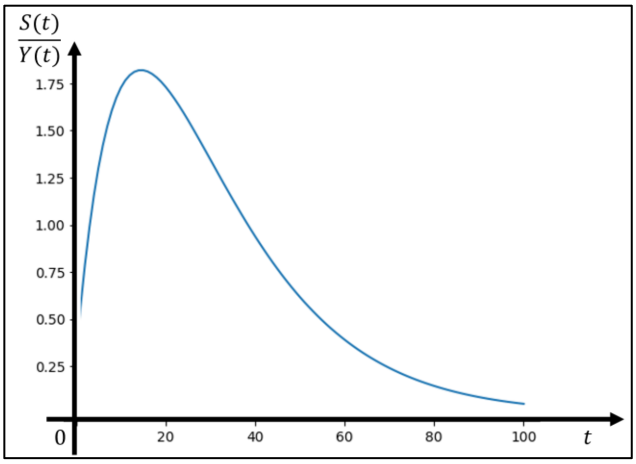


**Figure 2: A single-peaked path of the government-debt-to-GDP ratio ($r < d$ case).**

This numerical example corresponds to case (2) of Proposition 1 and illustrates that when $r < d$, the government-debt-to-GDP ratio can exhibit a temporary increase before converging in the long run, provided that the initial deficit-to-debt ratio is sufficiently large.

## DYNAMICS OF THE GOVERNMENT-DEBT-TO-GDP RATIO WHEN $r = d$

When $r = d$, the common ratio of the geometric series becomes unity. In this case, government debt evolves according to

$$S(t) = (1 + r)^t S_0 + t(1 + r)^{t-1} D_0.$$

Hence, the general expression for the government-debt-to-GDP ratio is given by

$$\frac{S(t)}{Y(t)} = \left(\frac{1 + r}{1 + g}\right)^t \left[\frac{S_0}{Y_0} + \frac{D_0}{Y_0}\frac{t}{1 + r}\right].$$

Differentiating $S(t)/Y(t)$ with respect to $t$yields

$$\left(\frac{1 + r}{1 + g}\right)^t \left[\left(\frac{S_0}{Y_0} + \frac{D_0}{Y_0}\frac{t}{1 + r}\right)\ln\left(\frac{1 + r}{1 + g}\right) + \frac{D_0}{Y_0(1 + r)}\right].$$

Since

$$\left(\frac{1 + r}{1 + g}\right)^t > 0,$$

the value of $t$ that sets the bracketed term equal to zero is given by

$$t^* = -\frac{S_0(1+r)}{D_0} - \frac{1}{\ln\left(\frac{1+r}{1+g}\right)}.$$

Therefore, the following relationship is established

$$t^* > 0 \iff \frac{S_0(1+r)}{D_0} < -\frac{1}{\ln\left(\frac{1+r}{1+g}\right)}.$$

Thus, we have the following poposition.

**Proposition 2**

**Suppose that** $g > r$**. If** $\frac{S_0(1+r)}{D_0} < \frac{1}{\ln\left(\frac{1+g}{1+r}\right)}$, then the government-debt-to-GDP ratio $S(t)/Y(t)$ converges to zero in the steady state, while increasing for an initial period.

**Numerical Example 2**

Let $r = 0.02$, $g = 0.08$, $S_0/Y_0 = 0.3$, and $D_0/Y_0 = 0.3$. The resulting time path of the government-debt-to-GDP ratio is as follows:

| ***t*** | **0** | **5** | **10** | **15** | **16** | **17** | **20** |
|---|---|---|---|---|---|---|---|
| $S(t)/Y(t)$ | 0.3000 | 1.3305 | 1.8301 | 1.9991 | 2.0059 | 2.0057 | 1.9710 |

In this case, $S(t)/Y(t)$ increases initially, reaches its maximum at $t = 16$, and then declines thereafter. Accordingly, the government-debt-to-GDP ratio follows a **single-peaked (unimodal) path**, as illustrated in **Figure 3**.

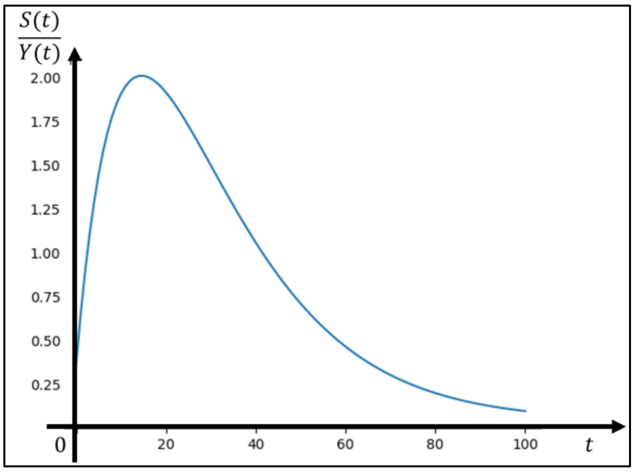


**Figure 3: A single-peaked path of the government-debt-to-GDP ratio ($r = d$ case).**

## CONCLUSION

This paper has examined the dynamic behavior of the government-debt-to-GDP ratio in an economy where interest rate, growth rate of fiscal deficit, and economic growth rate are constant over time. By explicitly modeling the joint dynamics of government debt, fiscal deficits, and GDP, the analysis derived closed-form expressions for the debt-to-GDP ratio and identified the conditions under which it converges, diverges, or follows a non-monotonic path.

The main analytical result is that even when the long-run sustainability condition $g>r$ and $g>d$ holds—ensuring convergence of the debt-to-GDP ratio to zero—the short- and medium-run dynamics need not be monotonic. In particular, the debt-to-GDP ratio may exhibit a single-peaked (unimodal) trajectory, increasing for an initial period before eventually declining. The paper derived explicit conditions under which such a temporary deterioration occurs and showed that these conditions depend crucially on the relative magnitudes of the interest rate, the growth rate of fiscal deficits, and the initial deficit-to-debt ratio.

Numerical examples were provided to illustrate each analytical case and to highlight the economic intuition behind the results. These examples confirm that focusing solely on long-run sustainability conditions can be misleading when assessing fiscal performance over finite horizons.

Overall, the findings suggest that a temporary rise in the government-debt-to-GDP ratio should not necessarily be interpreted as evidence of fiscal unsustainability. Instead, such behavior may be a natural outcome of transitional dynamics, even in economies that satisfy standard long-run sustainability conditions. This perspective has important implications for fiscal policy evaluation, emphasizing the need to distinguish between short-run dynamics and long-run fiscal solvency.

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